THINK by rephrasing if-then statements.

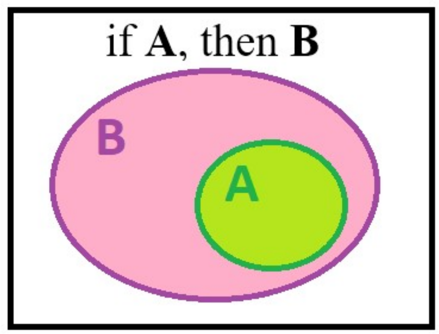
DEFINITION: An "*if-then* relational assertion" (also called a "material conditional") is a sentence that claims or suggests, "If one thing is true, then a second thing is true." Examples of *if-then* assertions are

- If you pay the bill on time, then you avoid a late fee.
- The auditorium is in the building. (Here, an *if-then* assertion is implied: "If you are in the auditorium, then you are in the building.")
- She falls asleep whenever you read a bedtime story. (In standard *if-then* form, this would be, "If you read a bedtime story, then she falls asleep.")
- All Martians are green. (In standard *if-then* form, "If you are a Martian, then you're green.")

When a *person* implies something, it means that they hint at it indirectly. This document, however, talks about when truth of some condition ("you pay your bills on time") implies truth of another condition ("you avoid a late fee"). In this context, "implies" means "leads to an inescapable conclusion," and it does not mean "cause." For example, the fact that *you are reading this sentence* implies (i.e., leads to an inescapable conclusion) that *you are alive*, but it certainly does not *cause* you to be alive!

Below, we show how to restate *if-then* assertions in a variety of "logically equivalent" ways so that truth of one reworded version implies truth of all versions. Falsehood of one implies falsehood of all. By mastering formal restatements of *if-then* assertions, you can better ascertain if they are true and select the restated version best suited to your communication goals.

Logic notation: Introduce symbols **A** and **B** (respectively called the *antecedent* and *consequent*) to stand for the two clauses in the *if-then* version of the statement. For example, if someone observes that *the auditorium is inside the building*, you might use



A = you are in the auditorium (antecedent)
B = you are in the building (consequent)

Symbols for negation:
notA = you are not in the auditorium
= you are outside the auditorium
notB = you are not in the building
= you are outside the building

The following 20 restatements of an *if-then* assertion are equivalent.

Whenever one of these 20 assertions is true, they are all true.
Whenever one of these 20 assertions is false, they are all false.

Active (**A→B**) and passive (**B←A**) forms	Contrapositives: **notB→notA** and **notA←notB**
If **A**, then **B**.	If **notB**, then **notA**.
A implies **B**.	**notB** implies **notA**.
A is not possible without **B**.	**notB** is not possible without **notA**.
A is sufficient for **B**.	**notB** is sufficient for **notA**.
B is necessarily implied by **A**.	**notA** is necessarily implied by **notB**.
A and **notB** can't be simultaneously true.	**notB** and **A** can't be simultaneously true.
B if/whenever **A**	**notA** if/whenever **notB**
B is implied by **A**.	**notA** is implied by **notB**.
B, because **A**.	**notA**, because **notB**.
B is necessary for **A**.	**notA** is necessary for **notB**.

In the last row, "**B** is necessary for **A**" can sound weird in sentence form. You can alternatively say, "**B** is necessarily true whenever **A** is true" or "for **A** to be true, it is necessary for **B** to be also true."

EXAMPLE: The auditorium is in the building. The following are all saying the same thing (and they're equivalent because truth of one implies truth of all others, or falsehood of one implies falsehood of all):

If you are in the auditorium, then you are in the building.	If you are not in the building, then you are not in the auditorium.
Being in the auditorium implies being in the building.	Being outside the building implies being outside the auditorium.
Being in the auditorium is not possible without being in the building.	Being outside the building is not possible without also being outside the auditorium.
Being in the auditorium is a sufficient condition for knowing you are in the building.	Being away from the building is sufficient for knowing you are also outside the auditorium.
Being in the building is necessarily implied by being in the auditorium.	Not being in the auditorium is necessarily implied by not being in the building.
It's impossible to be simultaneously in the auditorium but not the building.	It's impossible to be simultaneously outside the building and inside the auditorium.
You are in the building if you are in the auditorium.	You are not in the auditorium if you are not in the building.
Being in the building is implied by being in the auditorium.	Not being in the auditorium is implied by not being in the building.
You are in the building, because you are in the auditorium.	You must not be in the auditorium, because I happen to know that you are not in the building.
Being in the building is necessary for you to be in the auditorium. Your being in the building is necessarily true whenever you are in the auditorium. For you to be in the auditorium, it is necessary for you to be in the building.	Not being in the auditorium is necessary for you to be also away from the building. Your being outside the auditorium is necessarily true whenever you are outside the building. For you to be outside the building, it is necessary for you to be outside auditorium.

Once you recast a statement in *if-then* form, writing it in its 20 equivalent forms can guide how best to talk about it!

If the relationship **A➔B** is true, then the *contrapositive* (**notB➔notA**) is true.
If the relationship **A➔B** is false, then the *contrapositive* (**notB➔notA**) is false.
The relationship **A➔B** says nothing about whether or not the *converse* (**B➔A**) is true or false.
You can write **A➔➔B** if you wish to indicate that **A➔B**, but its converse **B➔A** is known to be false.
You can write **A🡐➔B** if you wish to indicate that **A➔B**, and its converse **B➔A** is known to be true.

The two-way relationship, **A🡐➔B** (meaning **A➔B** and **B🡐A**) can be stated in words as "**A** is necessary and sufficient for **B**" or "**A** iff **B**", where *iff* stands for "if and only if." Definitions must always provide a two-way relationship. Specifically, something can be called X *if and only if* it meets whatever conditions uniquely define X.

Potential confusion: "necessary and sufficient" vs. "necessarily sufficient"
The assertion "**A** is necessary and sufficient for **B**" means the arrow goes both ways (**A🡐➔B**). The similar-sounding assertion "**A** is necessarily sufficient for **B**" means there's some other—perhaps unstated— observation, **C**, that implies truth of the statement **A➔B**. In other words, **C➔**(**A➔B**).

Ruling out simultaneous truths: If someone says, "It's impossible for **S** and **T** to both be true!" (or, arcanely, **S** and **T** are disjoint), then they're making an *if-then* assertion **A➔B** in which **A**=**S** and **B**=**notT**. For example, if it's impossible for something to be both animal and vegetable, the corresponding *if-then* assertion would be "if it's an animal, then it's not a vegetable." If that's true, then so is the contrapositive: "if it's a vegetable, then it's not an animal." Nothing can be said about the converse: if it's not a vegetable, you can't conclude that it's an animal—it might be a mineral!

Negating if-then assertions. Consider the assertion "All men are pigs." If your first thought was "not my Nigel," then the proper way to disagree is to say, "Not all men are pigs." (Saying "all men are not pigs" would be a very different assertion that there is not one single man who is a pig).

EXERCISES

Fill in the following table with conditional assertions of the indicated forms ("the 20 ways").

A = you build it (alternatively, "you built it" or "the act of building it" or whatever works best)
B = they will come (or "they came" or whatever is appropriate to make a smooth sentence)

If **A**, then **B**. *If you build it, they will come.*	If **notB**, then **notA**.
A implies **B**.	**notB** implies **notA**.
A is not possible without **B**.	**notB** is not possible without **notA**.
A is sufficient for **B**.	**notB** is sufficient for **notA**.
B is necessarily implied by **A**.	**notA** is necessarily implied by **notB**.
A and **notB** can't be simultaneously true.	**notB** and **A** can't be simultaneously true.
B if **A**	**notA** if **notB**
B is implied by **A**.	**notA** is implied by **notB**.
B, because **A**.	**notA**, because **notB**.
B is necessary for **A**.	**notA** is necessary for **notB**.

Sorting fact from fiction. In the above list, your attempt at "**A** is not possible without **B**" might have ended up as something like "Building it is impossible unless they come." Is this a problem? We all know that you can build something whether someone comes or not. Recall that falsehood of one of "the 20" rephrased conditional assertions implies falsehood of all others. Consequently, the original assertion ("if you build it, they will come") would be *literally* false (yet still a wonderfully poetic *figurative* expression) or perhaps you didn't faithfully restate it. If you really want the statement to be literally true, then perhaps the improved version of "**A** is not possible without **B**" could be something like "Your having built it must never have happened if they didn't come." The main point is that these re-phrasings help to reveal possible issues with details, clarity, or truth of your assertions.

Broadly, unless all "20 ways" to rephrase are true, the original assertion is *literally* false. If so, then it is poetic, figurative, flat-out wrong, a generalization, wishful thinking, or ironic. Whenever you are unsure about the truth of an *if-then* assertion, write it in "the 20 ways" to understand it better. This rephrasing exercise might show you the best way to phrase your own if-then assertion, or it might show that your assertion is incomplete or otherwise flawed without further restriction/clarification to ensure that all 20 versions of it are clearly true.

EXERCISES

Which of the following sentences can be interpreted as conditional assertions (i.e., which of them correspond to *if-then* assertions, whether explicit or implicit)? For each conditional assertion, identify the antecedent, **A**, and consequent, **B**, for which **A➔B**. Are the assertions true? (Hint: if you aren't sure, try expressing them in the 20 alternative ways.)

1. If you are law abiding, then you pay taxes.
2. Eating cookies makes Jennie smile.
3. The cat slept on Jennie's lap.
4. This is the fourth sentence.
5. The word *fifth* means "after the first four."
6. All that glitters is not gold.
7. It's true that $x^2=4$ whenever $x = -2$.
8. Men are taller than women.
9. The cat purrs when you pet it.
10. If the earth were spinning, we'd fly off.
11. You can't have your cake and eat it too.
12. The World Cup is tomorrow.

Think of an *if-then* assertion that you believe is true. Fill out the following table and identify which versions make it seem "most obviously" true. Which versions are least effective at conveying truth?

A =
B =

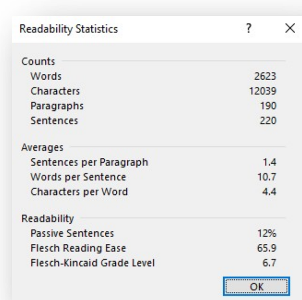
If **A**, then **B**.	If **notB**, then **notA**.
A implies **B**.	**notB** implies **notA**.
A is not possible without **B**.	**notB** is not possible without **notA**.
A is sufficient for **B**.	**notB** is sufficient for **notA**.
B is necessarily implied by **A**.	**notA** is necessarily implied by **notB**.
A and **notB** can't be simultaneously true.	**notB** and **A** can't be simultaneously true.
B if **A**	**notA** if **notB**
B is implied by **A**.	**notA** is implied by **notB**.
B, because **A**.	**notA**, because **notB**.
B is necessary for **A**.	**notA** is necessary for **notB**.

Think of an *if-then* assertion that you believe is false. Fill out the following table and identify which versions make it seem “most obviously” false. Which versions are least effective at suggesting falseness?

A =
B =

If **A**, then **B**.	If **notB**, then **notA**.
A implies **B**.	**notB** implies **notA**.
A is not possible without **B**.	**notB** is not possible without **notA**.
A is sufficient for **B**.	**notB** is sufficient for **notA**.
B is necessarily implied by **A**.	**notA** is necessarily implied by **notB**.
A and **notB** can’t be simultaneously true.	**notB** and **A** can’t be simultaneously true.
B if **A**	**notA** if **notB**
B is implied by **A**.	**notA** is implied by **notB**.
B, because **A**.	**notA**, because **notB**.
B is necessary for **A**.	**notA** is necessary for **notB**.

Generalizations are flawed thinking only when applied to individuals.
The assertion “seatbelts save lives” is a generalization. To write this as an *if-then* assertion, you need to refer to statistics. Perhaps say, “If you wear a seatbelt, then you are less likely to die in a wreck than an unbuckled person (with all other factors, such being sober, the same in the comparison).” Think of a TRUE generalization that often makes people shout indignantly, “Wait a minute, that’s not true of everyone!” Write it as an if-then assertion that refers to statistics in a way that scientists could test it objectively.



Think of a potentially inflammatory statement that many people believe to be true, but many others believe is false. Examples might be “Republicans are smarter than Democrats” or “If God exists, then he wouldn’t tolerate starving children.” Make it something that really causes people to gasp and recoil, labeling it as an inappropriate topic for the dinner table. Phrase it as an *if-then* assertion and write it in the 20 equivalent ways. Do not reveal if you personally believe the statement is true. Instead, identify which of the 20 restatements would appeal most to those who believe it is true. Likewise, identify which of the 20 ways would appeal to those who believe it is false. Using objective logic to analyze emotionally charged statements is an essential skill for any dispassionate critical thinker.

A =
B =

If **A**, then **B**.	If **notB**, then **notA**.
A implies **B**.	**notB** implies **notA**.
A is not possible without **B**.	**notB** is not possible without **notA**.
A is sufficient for **B**.	**notB** is sufficient for **notA**.
B is necessarily implied by **A**.	**notA** is necessarily implied by **notB**.
A and **notB** can’t be simultaneously true.	**notB** and **A** can’t be simultaneously true.
B if **A**	**notA** if **notB**
B is implied by **A**.	**notA** is implied by **notB**.
B, because **A**.	**notA**, because **notB**.
B is necessary for **A**.	**notA** is necessary for **notB**.

A MATH-GEEK EXAMPLE

A function $f(\mathbf{x})$ is called "linear" if and only if it has the property that for any values of a and b,

$$f(a\,\mathbf{x} + b\,\mathbf{y}) = a\,f(\mathbf{x}) + b\,f(\mathbf{y}).$$

As this holds for any a and b, it must also apply for the choice $a = b = 0$, thus giving the first entry in the table below. Complete the other entries in the table.

If **A**, then **B**. If $f(\mathbf{x})$ is linear, then $f(\mathbf{0}) = \mathbf{0}$.	If **notB**, then **notA**.
A implies **B**.	**notB** implies **notA**.
A is not possible without **B**.	**notB** is not possible without **notA**.
A is sufficient for **B**.	**notB** is sufficient for **notA**.
B is necessarily implied by **A**.	**notA** is necessarily implied by **notB**.
A and **notB** can't be simultaneously true.	**notB** and **A** can't be simultaneously true.
B if **A**	**notA** if **notB**
B is implied by **A**.	**notA** is implied by **notB**.
B, because **A**.	**notA**, because **notB**.
B is necessary for **A**.	**notA** is necessary for **notB**.

All three of the following functions are nonlinear. Use this table to prove it instantly for the first two, and explain why the table doesn't help to prove nonlinearity of the last function (sinusoid).

$f(\mathbf{x}) = m\mathbf{x}+\mathbf{b}$
$f(x) = \cos(x)$
$f(x) = \sin(x)$

Even though many people would call the first function linear, the correct term is *affine*.