Stability Analysis of Material Point Method

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Compressible Flow in One Dimension

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right] + \frac{\partial p}{\partial x} = 0$$

$$p = a^2\rho$$

Compressible Flow - Brackbill's Formulation

$$\frac{\partial \rho_1}{\partial t} + U_0 \frac{\partial \rho_1}{\partial x} + \rho_0 \frac{\partial u_1}{\partial x} = 0$$

$$\rho_0 \left[\frac{\partial u_1}{\partial t} + U_0 \frac{\partial u_1}{\partial x} \right] + \frac{\partial p_1}{\partial x} = 0$$

Substituting in $p_1 = a^2 \rho_1$ and rewriting the above equations in terms of their Lagrangian and Eularian parts. Note the Lagrangian particles move at constant velocity U_0.

$$\underbrace{\frac{D\rho}{Dt}}_{Lagrangian} = -\rho_0 \frac{\partial u_i}{\partial x}, \qquad \underbrace{\rho_0 \frac{Du}{Dt}}_{Lagrangian} = -a^2 \frac{\partial \rho_i}{\partial x}$$

J.U. Brackbill, The Ringing Instability in Particle-in-Cell Calculations of Low-Speed Flow, Journal of Computational Physics, 75, 469-492, 1988

The Algorithms

Formulation 1

1. $u_i^t = \sum_{p=1}^{np} u_p^t$
2. $\rho_i^t = \sum_{p=1}^{np} \rho_p^t$
3. $u_p^{t+1} = u_p^t + c\frac{dt}{h}(\rho_{i+1}^t - \rho_i^t)$
4. $\rho_p^{t+1} = \rho_p^t + c\frac{dt}{h}(u_{i+1}^t - u_i^t)$
5. $x_p^{t+1} = x_p^t + vdt$

Formulation 2

1. $\rho_i^t = \sum_{p=1}^{np} \rho_p^t$
2. $u_p^{t+1} = u_p^t + c\frac{dt}{h}(\rho_{i+1}^t - \rho_i^t)$
3. $u_i^{t+1} = \sum_{p=1}^{np} u_p^{t+1}$
4. $\rho_p^{t+1} = \rho_p^t + c\frac{dt}{h}(u_{i+1}^{t+1} - u_i^{t+1})$
5. $x_p^{t+1} = x_p^t + vdt$

Formulation 1 - Analysis

What happens at the nodes?

$$u_i^{t+1} = \sum_{p=1}^{np} S_{ip}^{t+1} u_p^{t+1}$$

Formulation 1 - Analysis

Substituting in u_p^{t+1} from the algorithm we get,

$$
\begin{aligned}
u_i^{t+1} &= \sum_{p=1}^{np} S_{ip}^{t+1} u_p^{t+1} \\
&= \sum_{p=1}^{np} S_{ip}^{t+1} [u_p^t + c\frac{dt}{h}(\rho_{i+1}^{t+1} - \rho_i^t)] \\
&= \sum_{p=1}^{np} S_{ip}^{t+1} u_p^t + c\frac{dt}{h} \sum_{p=1}^{np} S_{ip}^{t+1} (\rho_{i+1}^t - \rho_i^t) \\
&= u_i^t + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) u_p^t + c\frac{dt}{h} \sum_{p=1}^{np} S_{ip}^{t+1} (\rho_{i+1}^t - \rho_i^t).
\end{aligned}
$$

Formulation 1 - Analysis

If we apply the same steps to ρ_i^{t+1} and substitute we get the following set of equations Let the Courant term be $k = c\frac{dt}{h}$.

$$u_i^{t+1} = u_i^t + \sum_{p=1}^{np}(S_{ip}^{t+1} - S_{ip}^t)u_p^t + k\sum_{p=1}^{np} S_{ip}^{t+1}(\rho_{i+1}^t - \rho_i^t)$$

$$\rho_i^{t+1} = \rho_i^t + \sum_{p=1}^{np}(S_{ip}^{t+1} - S_{ip}^t)\rho_p^t + k\sum_{p=1}^{np} S_{ip}^{t+1}(u_{i+1}^t - u_i^t).$$

Formulation 1 - Analysis

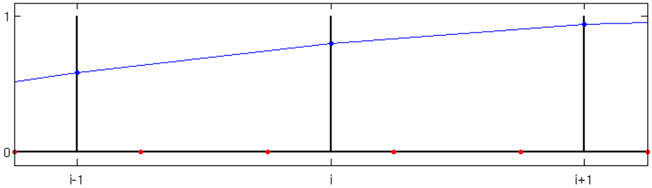
What can we say about the term

$$\sum_{p=1}^{np} S_{ip}^{t+1}(\rho_{i+1}^t - \rho_i^t)?$$

Formulation 1 - Analysis

Remember that

$$u_p^{t+1} = u_p^t + c\frac{dt}{h}(\rho_{i+1}^t - \rho_i^t).$$



Therefore

$$\sum_{p=1}^{np} S_{ip}^{t+1}(\rho_{i+1}^t - \rho_i^t) = C_1(\rho_i^t - \rho_{i-1}^t) + C_2(\rho_{i+1}^t - \rho_i^t)$$

$$C_1 + C_2 = 1$$

Formulation 1 - Analysis

If the particle velocity is zero then, $C_1 = C_2 = .5$ and $\sum_{p=1}^{np}(S_{ip}^{t+1} - S_{ip}^{t}) = 0$, and we get the following.

$$u_i^{t+1} = u_i^t + k(\rho_{i+1}^t - \rho_{i-1}^t)$$

$$\rho_i^{t+1} = \rho_i^t + k(u_{i+1}^t - u_{i-1}^t).$$

Using Von Neumann analysis we can gain some insight into the stability of the underlying schemes. Let ξ and η represent the errors at the nodes for u and ρ respectively.

$$\xi = aG^t e^{i\beta j}$$

$$\eta = bG^t e^{i\beta j}$$

Formulation 1 - Analysis

Expression of error growth at the nodes.

$$aG^{t+1}e^{i\beta j} = aG^{t}e^{i\beta j} + k(bG^{t}e^{i\beta j+1} - bG^{t}e^{i\beta j-1})$$

$$bG^{t+1}e^{i\beta j} = bG^{t}e^{i\beta j} + k(aG^{t}e^{i\beta j+1} - aG^{t}e^{i\beta j-1})$$

After rearranging of terms.

$$a(1 - G) + kb(e^{i\beta} - e^{-i\beta}) = 0$$

$$b(1 - G) + ka(e^{i\beta} - e^{-i\beta}) = 0$$

Formulation 1 - Analysis

The system of equations.

$$\begin{bmatrix} (1-G) & k(2i\sin\beta) \\ k(2i\sin\beta) & (1-G) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can use the fact that the determinant of the system is zero and set $\gamma = k(2i\sin\beta)$ we get the following.

$$(1-G)^2 - \gamma^2 = 0$$
$$G^2 - 2G + 1 - \gamma^2 = 0$$

Solving for G we get,

$$G = \frac{2 \pm \sqrt{4 - 4(1-\gamma^2)}}{2}$$
$$= 1 \pm \gamma.$$

This is an Unstable Scheme

Formulation 2 - Analysis

$$u_i^{t+1} = u_i^t + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) u_p^t + k \sum_{p=1}^{np} S_{ip}^{t+1} (\rho_{i+1}^t - \rho_i^t)$$

$$\rho_i^{t+1} = \rho_i^t + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) \rho_p^t + k \sum_{p=1}^{np} S_{ip}^{t+1} (\underline{u_{i+1}^{t+1} - u_i^{t+1}}).$$

Note the updated time step

Formulation 2 - Analysis

Again as in the previous formulation set particle velocity to zeros.

$$u_i^{t+1} = u_i^t + k(\rho_{i+1}^t - \rho_{i-1}^t)$$

$$\rho_i^{t+1} = \rho_i^t + k(u_{i+1}^{t+1} - u_{i-1}^{t+1}).$$

Formulation 2 - Analysis

Using Von Neumann analysis,

$$aG^{t+1}e^{i\beta j} = aG^{t}e^{i\beta j} + k(bG^{t}e^{i\beta j+1} - bG^{t}e^{i\beta j-1})$$

$$bG^{t+1}e^{i\beta j} = bG^{t}e^{i\beta j} + k(aG^{t+1}e^{i\beta j+1} - aG^{t+1}e^{i\beta j-1})$$

After rearranging of terms.

$$a(1-G) + kb(e^{i\beta} - e^{-i\beta}) = 0$$

$$b(1-G) + kaG(e^{i\beta} - e^{-i\beta}) = 0$$

Formulation 2 - Analysis

The system of equations.

$$\begin{bmatrix} (1-G) & k(2i\sin\beta) \\ kG(2i\sin\beta) & (1-G) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve the determinant of the system and set $\gamma = k(2i\sin\beta))$.

$$(1-G)^2 - G\gamma^2 = 0$$

$$G^2 - G(\gamma^2 + 2) + 1 = 0$$

Solving for G we get,

$$G = \frac{(\gamma^2 + 2) \pm \sqrt{(\gamma^2 + 2)^2 - 4}}{2}$$

$$= 1 + \frac{\gamma^2 \pm \sqrt{\gamma^4 + 2\gamma^2}}{2}$$

Formulation 2 - Analysis

Given $\gamma = k(2i \sin \beta)$ what do we learn form G?

$$G = 1 + \frac{\gamma^2 \pm \sqrt{\gamma^4 + 2\gamma^2}}{2}$$

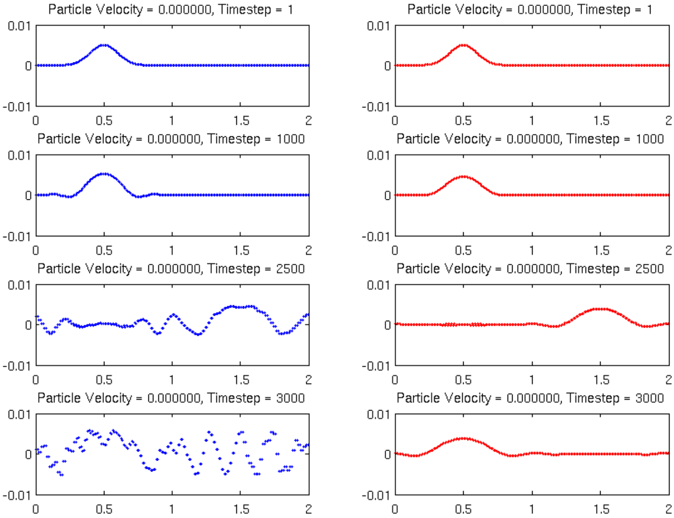
$$= 1 - 2k^2 \sin^2 \beta \pm \frac{\sqrt{16k^2 \sin^4 \beta - 8k^2 \sin^2 \beta}}{2}$$

$$= 1 - 2k^2 \sin^2 \beta \pm k \sin \beta \sqrt{4k^2 \sin^2 \beta - 2}$$

If $k \leq \frac{1}{\sqrt{2}}$

$$|G| = \sqrt{1 - 2k^2 \sin^2 \beta}$$

Example 1



Particle Velocity = 0.000000, Timestep = 1
Particle Velocity = 0.000000, Timestep = 1
Particle Velocity = 0.000000, Timestep = 1000
Particle Velocity = 0.000000, Timestep = 1000
Particle Velocity = 0.000000, Timestep = 2500
Particle Velocity = 0.000000, Timestep = 2500
Particle Velocity = 0.000000, Timestep = 3000
Particle Velocity = 0.000000, Timestep = 3000
0.01
0
-0.01
0
0.5
1
1.5
2

1d Linear Elastic Model

Starting with the Cauchy Momentum Equation

$$\rho \frac{Dv}{Dt} = \nabla \cdot \sigma + b$$

Rewriting for the 1d form and substituting for σ and neglecting the body force.

$$\rho \frac{Dv}{Dt} = \frac{\partial}{\partial x} \left(E \frac{\partial u}{\partial x} \right)$$

Given ρ and E as constants we get, $c = \sqrt{E/\rho}$,

$$\frac{Dv}{Dt} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

1d Linear Elastic Model

The Coupled Set of Equations forming the Wave Equation

$$\left.\begin{aligned} \frac{Du}{Dt} &= v \\ \frac{Dv}{Dt} &= c^2 \frac{\partial^2 u}{\partial x^2} \end{aligned}\right\} \Longrightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The Algorithm

1. $u_i^t = \sum_{p=1}^{np} S_{ip} u_p^t$
2. $v_i^t = \sum_{p=1}^{np} S_{ip} v_p^t$
3. $a_i^t = \frac{c^2}{h^2}(u_{i+1}^t - 2u_i^t + u_{i-1}^t)$
4. $v_i^{t+1} = v_i^t + dt * a_i^t$
5. $v_p^{t+1} = v_p^t + dt \sum_{p=1}^{np} S_{ip} a_i^t$
6. $u_p^{t+1} = u_p^t + dt \sum_{p=1}^{np} S_{ip} v_i^{t+1}$

Analysis

As we did in the previous formulations lets see what happens at the nodes for u_i.

$$u_i^{t+1} = \sum_{p=1}^{np} S_{ip}^{t+1} u_p^{t+1}.$$

Analysis

Substituting in for u_p^{t+1} from the above algorithm we get the following.

$$
\begin{aligned}
u_i^{t+1} &= \sum_{p=1}^{np} S_{ip}^{t+1} u_p^{t+1} \\
&= \sum_{p=1}^{np} S_{ip}^{t+1} (u_p^t + dt \sum_{p=1}^{np} S_{ip}^t v_i^{t+1}) \\
&= \sum_{p=1}^{np} S_{ip}^{t+1} u_p^t + dt \sum_{p=1}^{np} S_{ip}^{t+1} \sum_{p=1}^{np} S_{ip}^t v_i^{t+1} \\
&= u_i^t + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) u_p^t + dt \sum_{p=1}^{np} S_{ip}^{t+1} \sum_{p=1}^{np} S_{ip}^t v_i^{t+1}.
\end{aligned}
$$

Analysis

If the following condition holds

$$\sum_{p=1}^{np} S_{ip} = 1,$$

then we get the following

$$u_i^{t+1} = u_i^t + \sum_{p=1}^{np}(S_{ip}^{t+1} - S_{ip}^t)u_p^t + dt\sum_{p=1}^{np} S_{ip}^{t+1}\sum_{p=1}^{np} S_{ip}^t v_i^{t+1}$$

$$= u_i^t + \sum_{p=1}^{np}(S_{ip}^{t+1} - S_{ip}^t)u_p^t + v_i^{t+1}dt.$$

Analysis

Now lets look at v_i.

$$
\begin{aligned}
v_i^{t+1} &= \sum_{p=1}^{np} S_{ip}^{t+1} v_p^{t+1} \\
&= \sum_{p=1}^{np} S_{ip}^{t+1} (v_p^t + dt \sum_{p=1}^{np} S_{ip}^t a_i^t) \\
&= v_i^t + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) v_p^t + dt \sum_{p=1}^{np} S_{ip}^{t+1} \sum_{p=1}^{np} S_{ip}^t a_i^t \\
&= v_i^t + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) v_p^t + a_i^t dt.
\end{aligned}
$$

Analysis

Now substitute in for a_i

$$\begin{aligned} v_i^{t+1} &= v_i^t + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) v_p^t + a_i^t dt \\ &= v_i^t + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) v_p^t + \frac{c^2 dt}{h^2} (u_{i+1}^t - 2u_i^t + u_{i-1}^t) \end{aligned}$$

Analysis

We can now combine the two equations, u_i^{t+1} and v_i^{t+1}.

$$u_i^{t+1} = u_i^t + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) u_p^t + v_i^{t+1} dt$$

$$= u_i^t + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) u_p^t + v_i^t dt + \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t) v_p^t + a_i^t dt^2$$

Knowing

$$u_i^t = u_i^{t-1} + \sum_{p=1}^{np} (S_{ip}^t - S_{ip}^{t-1}) u_p^{t-1} + v_i^t dt$$

we can arrive at $v_i^t dt$

$$v_i^t dt = u_i^t - u_i^{t-1} - \sum_{p=1}^{np} (S_{ip}^t - S_{ip}^{t-1}) u_p^{t-1}$$

Analysis

After substituting for v_i^t and rearranging some terms we get the following finite difference scheme at the nodes,

$$\frac{u_i^{t+1} - 2u_i^t + u_i^{t-1}}{dt^2} = c^2 \frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{h^2} + S,$$

where

$$S = \frac{1}{dt^2} \sum_{p=1}^{np} (S_{ip}^{t+1} - S_{ip}^t)(u_p^t + v_p^t dt) - (S_{ip}^t - S_{ip}^{t-1}) u_p^{t-1}.$$

Analysis

Von Neumann analysis gives us the following.

$$G^2 - 2G\gamma + 1 = 0, \qquad \gamma = 1 - 2k^2 \sin^2(\beta/2)$$

$$G = \gamma \pm \sqrt{\gamma^2 - 1}$$

If $k \leq 1$ then $|\gamma| \leq 1$ and $\sqrt{1-\gamma^2}$ is real for all β and then we get,

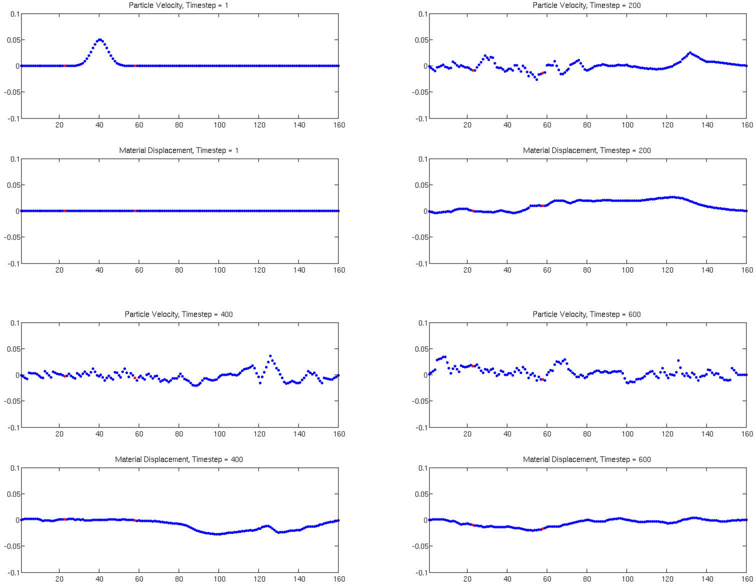
$$G = \gamma \pm i\sqrt{1-\gamma^2}$$

$$|G| = \sqrt{\gamma^2 + (1-\gamma^2)}$$

$$= 1.$$

John Noye, Finite Difference Techniques for Partial Differential Equations, Computational Techniques for Differential Equations, Elsevier Science Publishers B.V., p.285, 1984

Example 2


Particle Velocity, Timestep = 1
Particle Velocity, Timestep = 200
Material Displacement, Timestep = 1
Material Displacement, Timestep = 200
Particle Velocity, Timestep = 400
Particle Velocity, Timestep = 600
Material Displacement, Timestep = 400
Material Displacement, Timestep = 600

Conclusions and Further Explorations

1. The underlying formulation matters when it comes to stability.
2. How does the source term contribute to stability?
3. How are ringing instability and formulation instability connected?