Slides of Lecture 12

Today's Class:
Review of Eigen-Analysis
Solution to Homework of Lecture 11
More Eigen-Analysis

Recall

In Lecture 11, we found that the governing equations $M\ddot{x} + Kx = F$ are diagonalized by the transformation $x = P\beta$, where the columns of P are the eigenvectors v_n solving $(-\omega_n^{\;2}M + K)v_n = 0$.

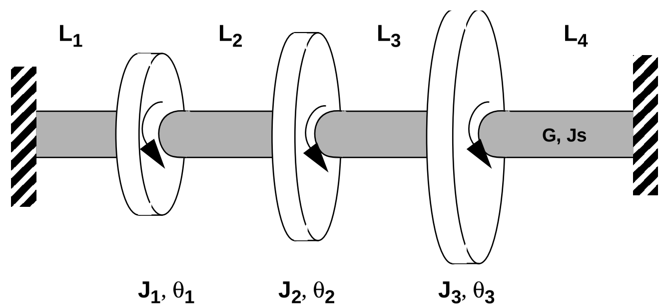
Those eigenvectors are mutually orthogonal with respect to the Mass and Stiffness matrices. Further, we scale those columns so that they are orthonormal with respect to the mass matrix:

$$P^T M P = I$$

In that case, $P^T K P = \begin{bmatrix} \ddots & & \\ & \omega_n^2 & \\ & & \ddots \end{bmatrix}$

When a structure is correctly modeled by a finite number of discrete degrees of freedom, then $\omega_0^2 = min((x^T K x)/(x^T M x))$

Homework
Due Today



Lets consider a structure similar to the disk-shaft system in Meirovitch.

Ignoring the kinetic energy of the shaft:

- **Derive the linear equations for this system**
- **Write the matrix eigen-equation**
- **Assuming that** $J_1 = J_2 = J_3$ **and that** $L_1 = L_2 = L_3 = L_4$**, write a dimensionless version of the matrix eigen equation.**
- **Using Matlab, find the eigen solutions.**
- **Plot modes as in Fig 4.1.**
- **Write the modal matrix. Remember to normalize w.r.t. mass matrix.**

Homework Solution

In Lecture 10, we derived stiffness matrix from the potential energy

$$V = \frac{1}{2} G J_s \left[\frac{\theta_1^2}{L_1} + \frac{(\theta_2 - \theta_1)^2}{L_2} + \frac{(\theta_3 - \theta_2)^2}{L_3} + \frac{\theta_3^2}{L_4} \right].$$

$$[K_{ij}] = \left[\frac{\partial^2 V}{\partial\theta_i \partial\theta_j} \right] = G J_s \begin{bmatrix} \frac{1}{L_1} + \frac{1}{L_2} & -\frac{1}{L_2} & 0 \\ -\frac{1}{L_2} & \frac{1}{L_2} + \frac{1}{L_3} & -\frac{1}{L_3} \\ 0 & -\frac{1}{L_3} & \frac{1}{L_3} + \frac{1}{L_4} \end{bmatrix}$$

Note that J_s is a polar moment of inertia of the area ($\pi R^4 / 2$). The units of polar moment of inertia are Length^4.

Homework Solution

We derived the mass matrix from the kinetic energy

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}J_3\dot{\theta}_3^2.$$

$$[M_{ij}] = \left[\frac{\partial^2 T}{\partial\dot{\theta}_i\partial\dot{\theta}_j}\right] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$$

Here J_1, J_2, and J_3 are mass moments of inertia. For a disk, $J_M = \rho J_A \Delta H$ where ρ is mass density, J_A is polar moment of inertia, and ΔH is the disk thickness. The units of J_M is are Mass*Length^2.

Homework Solution

The governing equation is

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + GJ_s \begin{bmatrix} \frac{1}{L_1} + \frac{1}{L_2} & -\frac{1}{L_2} & 0 \\ -\frac{1}{L_2} & \frac{1}{L_2} + \frac{1}{L_3} & -\frac{1}{L_3} \\ 0 & -\frac{1}{L_3} & \frac{1}{L_3} + \frac{1}{L_4} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = 0$$

Homework Solution

Assuming that $L_1 = L_2 = L_3 = L_4 = L$ **and that**

$J_1 = J_2 = J_3 = J_M$,

$$J_M \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \frac{GJ_s}{L} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = 0$$

The corresponding algebraic eigen-problem is

$$-\omega^2 J_M \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \frac{GJ_s}{L} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = 0$$

Homework Solution

There is only one dimensionless group to form: Let $\alpha^2 = \omega^2 \dfrac{J_M L}{G J_s}$.

Our algebraic equation is now

$$\left(-\alpha^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \right) \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = 0$$

The eigen-solutions are

$$\left(2 - \sqrt{2}, \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \right), \left(2, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right), \left(2 + \sqrt{2}, \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix} \right)$$

Homework Solution

The MODAL MATRIX is $P = \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & (-1/\sqrt{2}) \\ 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix}$

Test diagonalization: $P^T M P = P^T I P = I$

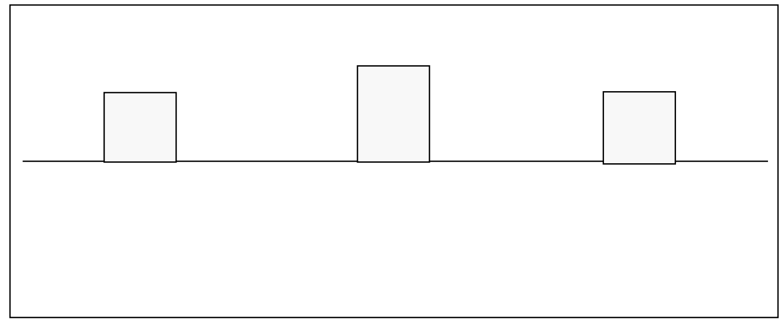
and $P^T K P = \begin{bmatrix} 2-\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2+\sqrt{2} \end{bmatrix}$

Homework Solution

Mode Shapes

Mode 1

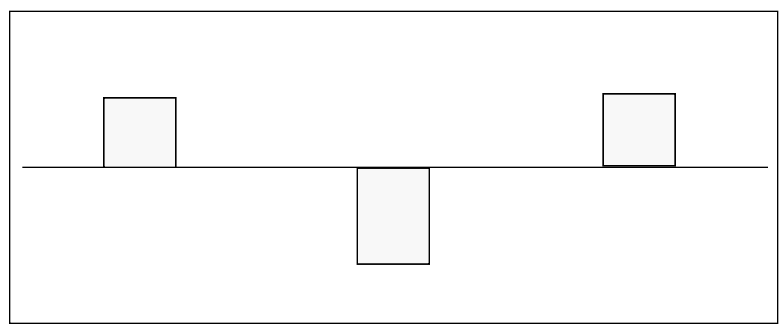
Lowest Energy



Mode 2

Mode 3

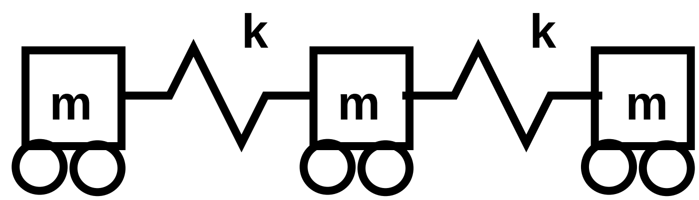
Highest Energy



Systems with Rigid Body Modes

Because we know that the modes are orthogonal with respect to the mass matrix, we may use known eigenvectors to reduce the size of the remaining eigen-problem. This is particularly useful for systems with easily recognized rigid body modes.

Consider the system shown. The



mass matrix is $M = \begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix}$

and the stiffness matrix is $K = \begin{bmatrix} k & -k & \\ -k & 2k & -k \\ & -k & k \end{bmatrix}$**. We see right away that a**

rigid body mode is $v_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$**. (Observe** $Kv_1 = 0$**.)**

Systems with Rigid Body Modes

We know that there are two remaining eigen modes and that they are orthogonal to the rigid body mode:

$v_1^T M x = m \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x = m(x_1 + x_2 + x_3) = 0$**. This provides a constraint that can be used to reduce the order of the system.**

$x_2 = -(x_1 + x_3)$ **defines a transformation matrix**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = Tx.$$

Systems with Rigid Body Modes

Resulting in $\bar{M} = T^T M T = m\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ **and** $\bar{K} = T^T K T = k\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

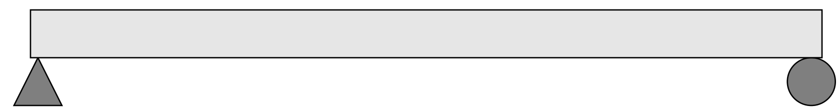
The eigenvectors of this reduced system are

$$\bar{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow v_2 = T\bar{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$ **and**

$$\bar{v}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow v_3 = T\bar{v}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Eigen Example
Another Assumed Modes Problem

Lets consider a simply supported beam. We shall approximate the deformed shape by a low-order polynomial satisfying the geometric boundary conditions $y(0, t) = y(L, t) = 0$:



$$y(x, t) = \sum_{k=1}^{N} A_k(t)\left(\frac{x}{L}\right)^k\left(1 - \frac{x}{k}\right) = \sum_{k=1}^{N} A_k(t) f_k(x)$$

We shall use the $A_k(t)$ as generalized coordinates and solve for the eigenmodes and frequencies in terms of them.

It is important to note that these polynomials can at best be approximations for the vibration modes of this structure.

Eigen Example
Mass Matrix

The kinetic energy is

$$T = \int_0^L \frac{m}{2}\dot{y}^2 dt = \frac{mL}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\dot{A}_i\dot{A}_j\int_0^1 f_i(sL)f_j(sL)ds$$

$$= \frac{mL}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\dot{A}_i\dot{A}_j\frac{2}{(i+j+1)(i+j+2)(i+j+3)}$$

$$= (mL)\dot{A}^T\overline{M}\dot{A}$$

where $\overline{M}_{ij} = \dfrac{2}{(i+j+1)(i+j+2)(i+j+3)}$

Eigen Example
Mass Matrix

The potential energy is

$$V = \int_0^L \frac{EI}{2} y''^2 dt$$

$$= \frac{EI}{2L^4} \sum_{i=1}^{N} \sum_{j=1}^{N} A_i A_j \int_0^1 \left(\frac{\partial^2 f_i}{\partial s^2}(sL) \right) \left(\frac{\partial^2 f_j}{\partial s^2}(sL) \right) ds$$

$$= \frac{EI}{2L^4} A^T \overline{K} A$$

where $\overline{K_{ij}} = \dfrac{2ij(i^2 + j^2 - 6(i+j) + 3ij + 7)}{(i+j-1)(i+j-2)(i+j-3)}$ **for** $i + j > 1$

and $\overline{K}_{11} = 4$

Eigen Example
Lagrange Equation

The governing equation is

$$(mL)\overline{M}\ddot{A} + \left(\frac{EI}{L^3}\right)\overline{K}A = 0$$

The corresponding algebraic eigenvalue is achieved by postulating that $A(t) = Re\{A_0 e^{i\omega t}\}$ **problem is**

$$\left(-(\omega^2 mL)\overline{M} + \left(\frac{EI}{L^3}\right)\overline{K}\right)A_0 = 0$$

We nondimensionalize this by setting $\alpha^2 = \dfrac{\omega^2 (mL) L^3}{EI}$ **and the eigen problem is now** $(-\alpha^2 \overline{M} + \overline{K})A_0 = 0$

Eigen Example
Case N=1

$$\overline{M}^{1} = \left[0.0333\right], \overline{K}^{1} = \left[4.000\right].$$

Eigen solution $\alpha_1^2 = 120$ **and** $P_1 = [5.4772]$

This asserts that the frequency $\alpha_1^2 = 120$ **is associated with the postulated shape** $y(x) = \left(\frac{x}{L}\right)\left(1 - \frac{x}{L}\right)$.

From Lecture 11, we know that this is an upper bound for the square of the first natural frequency.

Case N=3

$$\overline{M}^3 = \begin{bmatrix} 0.0333 & 0.0167 & 0.0095 \\ 0.0167 & 0.0095 & 0.0060 \\ 0.0095 & 0.0060 & 0.0040 \end{bmatrix}, \quad \overline{K}^3 = \begin{bmatrix} 4.0000 & 2.0000 & 2.0000 \\ 2.0000 & 4.0000 & 4.0000 \\ 2.0000 & 4.0000 & 4.8000 \end{bmatrix}$$

Solution:
$$\alpha_1^2 = 97.47, \quad \alpha_2^2 = 2520, \quad \alpha_3^2 = 17375 \quad , \quad P^3 = \begin{bmatrix} 4.4458 & -14.4914 & 28.6397 \\ 4.7965 & 28.9828 & -132.7290 \\ -4.7965 & -0.0000 & 132.7290 \end{bmatrix}$$

This asserts that $\omega_1^2 = 97.47$ is associated with the shape

$$y(x) = \left(4.45\frac{x}{L} + 4.80\left(\frac{x}{L}\right)^2 - 4.80\left(\frac{x}{L}\right)^3\right)\left(1 - \frac{x}{L}\right)$$

and is still an upper bound for the square of the first natural frequency.

Convergence with Of Eigenvalues in Assumed Modes Problem

It is interesting to examine how the calculated natural frequency

Table 1:

Terms	α_1^2	α_2^2	α_3^2	α_4^2	α_5^2
1	120				
2	120	2520			
3	97.47	2520	17400		
4	97.47	1572	17370	76188	
5	97.41	1571.8	8161	76188	258502
Exact	97.409	1558.5	7890	24937	60880

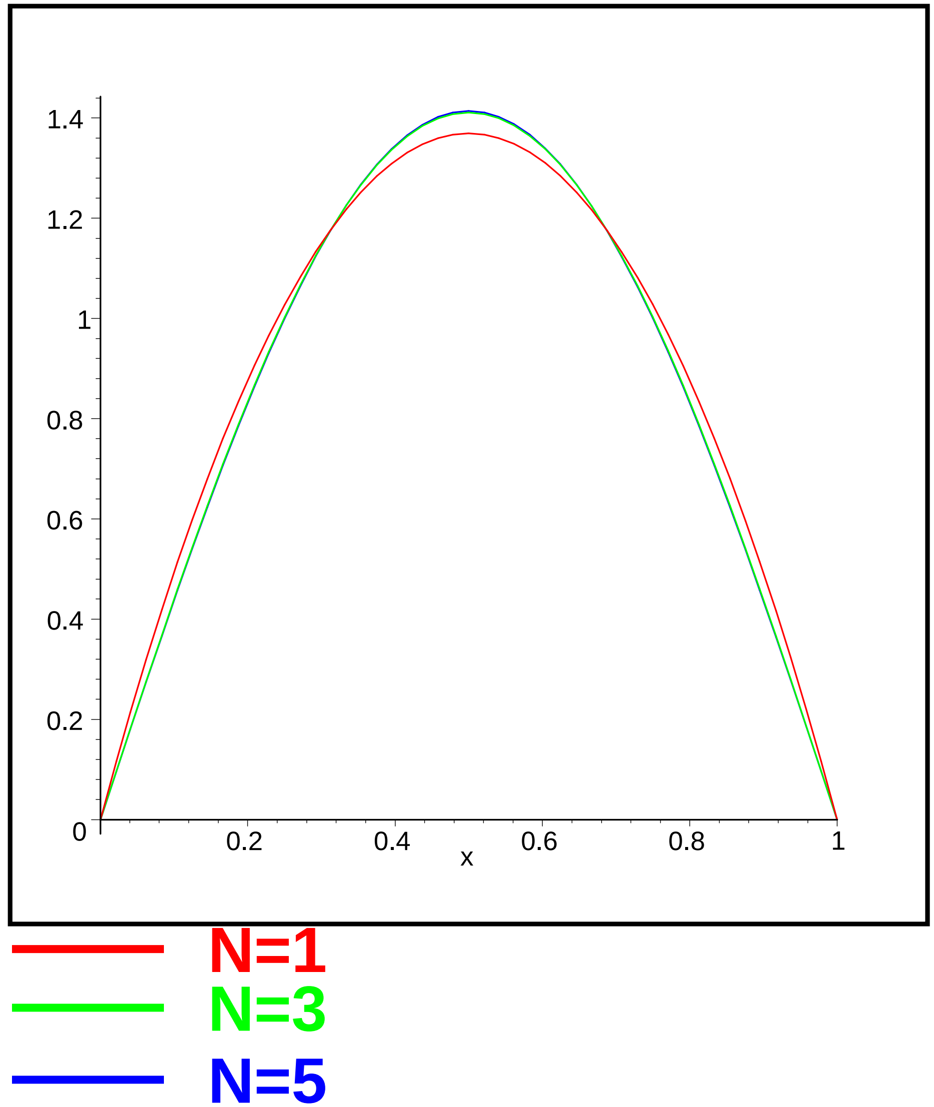
depends on the number of assumed modes. There is a reason for this that we will explore soon.

Note that even with the lowest order guess for the mode shape, we were able to get a reasonable estimate for the first natural frequency.

Mode Shapes: Look at Approximation for the First Mode for Each Order

As more polynomials are admitted, the computed mode shape appears to converge on the exact solution. In this case the exact solution is

$$y(x) = \sin\left(\pi \frac{x}{L}\right)$$



An Important Distinction About Eigenmodes and Eigenvectors

In the above, $X = \begin{bmatrix} \mathbf{4.4458} \\ \mathbf{4.7965} \\ \mathbf{-4.7965} \end{bmatrix}$ **is an exact eigenvector of the above algebraic eigen problem. (MATLAB says so!).**

BUT

$$y(x) = \left(4.45\frac{x}{L} + 4.80\left(\frac{x}{L}\right)^2 - 4.80\left(\frac{x}{L}\right)^3\right)\left(1 - \frac{x}{L}\right)$$ **is very much NOT the exact first mode of the simply supported beam, even though it is the best approximation that can be achieved with a quartic polynomial.**

A More General Principle
Rayleigh Quotient

For any elastic system, the lowest natural frequency is achieved by that displacement field that minimizes the ratio of potential to kinetic energy,

$$\lambda = \min_{T((y) > 0)} \left(\frac{V(y)}{T(y)} \right)$$

In the above, the field y is used as both the displacement and the velocity field.

The above relation was already proven for cases of point masses, but holds true for continuous systems. The minimization is over all displacement fields which are square integrable and sufficiently differentiable.

Restriction to a smaller set of displacement fields will yield an upper-bound for the first eigen value.

We shall explore this more as we focus on continuous systems later on.

Next Time

Post-Mortem on MidTerm
Modal Damping
Modal Strain Energy
Transfer Functions