Lecture 13

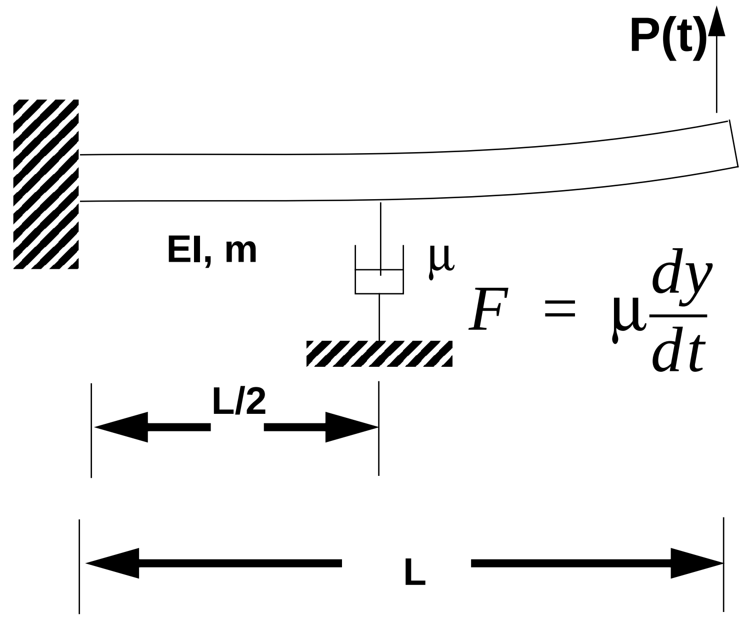
TODAY

Solutions To Midterm

Eigen Solutions to the Differential Equations
Damping Matrices
Modal Damping

Mid-Term Take Home Exam
Problem 1. Assumed Modes

Consider the cantilevered Beam shown here. There is a time-dependent load applied at the end and a dashpot placed at the center of the span.



Assume a single deformation mode:

$$y(x, t) = A_1(t)\left(\frac{x}{L}\right)^2.$$

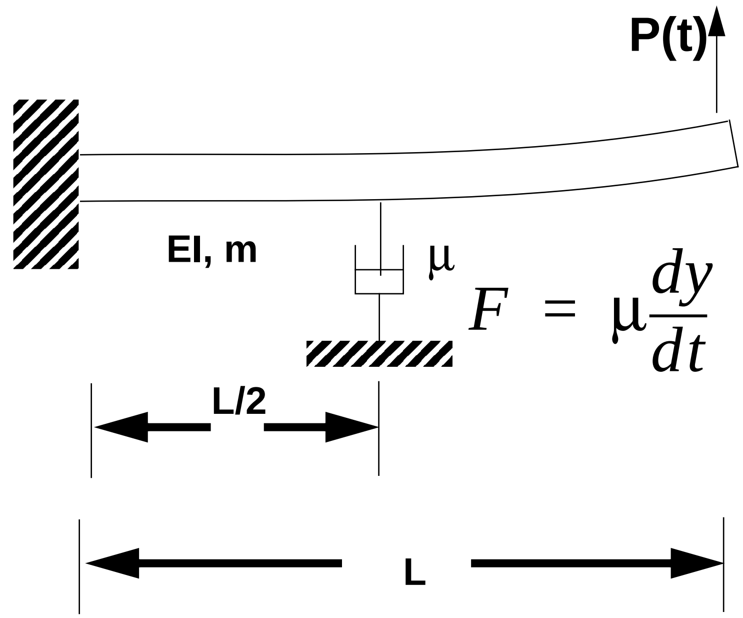
- **Derive the governing equation for $A_1(t)$.**
- **Assuming that P is harmonic, calculate the *complex* magnification factor for the tip displacement as a function of the frequency.**
- **Calculate the time-averaged rate of energy dissipation as a function of frequency and force amplitude.**

Solution to Problem

$$y(x,t) = A_1(t)\left(\frac{x}{L}\right)^2.$$

$$T = \int_0^L \frac{m}{2}\dot{A}^2\left(\frac{x}{L}\right)^4 dx = \frac{Lm}{2\cdot 5}\dot{A}^2$$

$$V = \int_0^L \frac{EI}{2}A^2\left(\frac{2}{L^2}\right)^2 dx = \frac{4EI}{2L^3}A^2$$



$$F = \mu\frac{dy}{dt}$$

Solution to Problem

Applied force: $\delta W = Q_P \delta A = P \delta y(L) \Rightarrow Q_P = P$

Damping force:

$$\delta W = Q_D \delta A = F_D \delta y\left(\frac{L}{2}\right) = -\mu\left(\frac{\dot{A}}{4}\right)\frac{\delta A}{4} \Rightarrow Q_D = -\frac{\mu \dot{A}}{16}$$

The Lagrange equation is $\dfrac{Lm}{5}\ddot{A} + \dfrac{\mu \dot{A}}{16} + \dfrac{4EI}{L^3}A = P(t)$

In more familiar terms, this is $M\ddot{A} + C\dot{A} + KA = P$

where $M = \dfrac{Lm}{5}$, $C = \dfrac{\mu}{16}$, **and** $K = \dfrac{4EI}{L^3}$

Solution to Problem

Assume that $P(t) = Re\{P_0 e^{i\omega t}\}$**, then we expect the resulting displacement will be** $A(t) = Re\{A_0 e^{i\omega t}\}$**. Substitution into the governing equation yields**

$$Re\{([-\omega^2 M + i\omega C + K]A_0 - P_0)e^{i\omega t}\} = 0$$

from which we conclude that $[-\omega^2 M + i\omega C + K]A_0 = P_0$

which is re-arranged to $\left[1 - \left(\frac{\omega}{\omega_0}\right)^2 + 2i\zeta\frac{\omega}{\omega_0}\right]A_0 = \frac{P_0}{K}$

where $\omega_0^2 = \frac{K}{M} = \frac{20EI}{mL^4}$ **and** $\zeta = \frac{C}{2M\omega_0} = \frac{\mu L}{64}\sqrt{\frac{5}{mEI}}$

Solution to Problem

Solving for A_0 in terms of P_0,

$$A_0 = \frac{P_0}{K} H(\omega)$$

where $H(\omega) = \dfrac{1}{\left[1 - \left(\dfrac{\omega}{\omega_0}\right)^2 + 2i\zeta\dfrac{\omega}{\omega_0}\right]}$ **is the Magnification Factor.**

Solution to Problem

The average rate of energy dissipation is

$$D = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} P(t)\dot{y}(L,t)\,dt$$

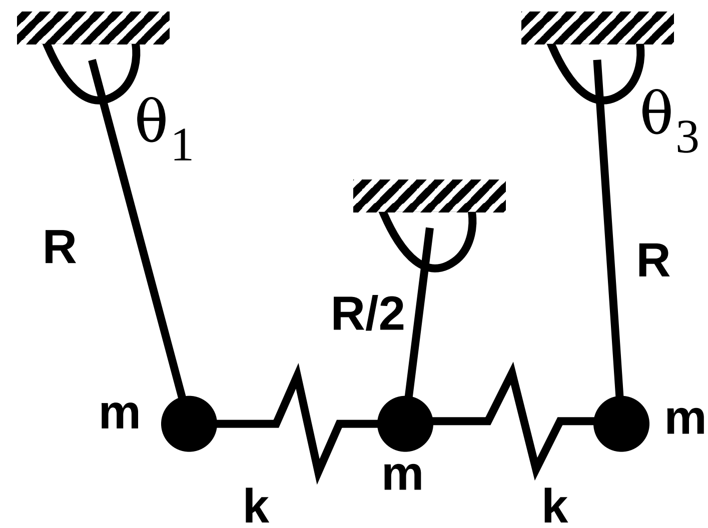
$$= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} Re\{P_0 e^{i\omega t}\} Re\{i\omega A_0 e^{i\omega t}\}\,dt$$

$$= \frac{1}{2} Re\{(P_0{}^*)(i\omega A_0)\} = \frac{1}{2} Re\{(P_0{}^*)(i\omega P_0 H(\omega)/K)\}$$

$$= \omega\frac{L^3|P_0|^2}{8EI} Re\{iH(\omega\} = -\omega\frac{L^3|P_0|^2}{8EI} Im\{H(\omega\}$$

Problem 2.

Consider the three pendula shown.



- **Calculate the governing equations for the three angles θ_1, θ_2, & θ_3 in terms of the parameters shown. (Ignore vertical components of spring extension.)**
- **Linearize the equations, evaluating the mass and stiffness matrices.**
- **Impose the constraint that $\theta_1 = \theta_3$, writing the constraint matrix. Write the mass and stiffness matrices in the reduced system.**
- **Calculate the eigenmodes and frequencies in the reduced system.**
- **Express those eigenmodes in terms of the full set of displacement degrees of freedom, θ_1, θ_2, & θ_3.**

Solution

Derivation of governing equations:

$$V = \frac{k}{2}\left[\left(\frac{R}{2}\right)\sin\theta_2 - R\sin\theta_1\right]^2 + \frac{k}{2}\left[\left(\frac{R}{2}\right)\sin\theta_2 - R\sin\theta_3\right]^2$$
$$-mgR\cos\theta_1 - mg\frac{R}{2}\cos\theta_2 - mgR\cos\theta_3 \quad .$$

$$\frac{\partial V}{\partial\theta_1} = mgR\theta_1 + kR^2\left(\theta_1 - \frac{\theta_2}{2}\right)$$

$$\frac{\partial V}{\partial\theta_2} = mg\frac{R}{2}\theta_2 + \frac{kR^2}{2}(\theta_2 - \theta_1 - \theta_3)$$ **and**

$$\frac{\partial V}{\partial\theta_3} = mgR\theta_3 + kR^2\left(\theta_3 - \frac{\theta_2}{2}\right)$$ **after linearization.**

Solution

$$T = \frac{m}{2}R^2\dot{\theta}_1^2 + \frac{m}{2}\left(\frac{R}{2}\right)^2\dot{\theta}_2^2 + \frac{m}{2}R^2\dot{\theta}_3^2$$

Yielding the governing equations

$$\frac{R}{g}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & & 1 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} (1+\gamma) & -\gamma/2 & 0 \\ -\gamma/2 & (1+\gamma)/2 & -\gamma/2 \\ 0 & -\gamma/2 & (1+\gamma) \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = 0$$

where $\gamma = (kR)/(mg)$

Solution

Setting $\begin{bmatrix}\theta_1 & \theta_2 & \theta_3\end{bmatrix}^T = Re\left\{e^{i\omega t}\begin{bmatrix}A_1 & A_2 & A_3\end{bmatrix}^T\right\}$

The resulting algebraic eigen problem becomes

$$-\alpha^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & & 1\end{bmatrix}\begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix} + \begin{bmatrix}(1+\gamma) & -\gamma/2 & 0\\ -\gamma/2 & (1+\gamma)/2 & -\gamma/2\\ 0 & -\gamma/2 & (1+\gamma)\end{bmatrix}\begin{bmatrix}A_1\\ A_2\\ A_3\end{bmatrix} = 0$$

where $\alpha^2 = R\omega^2/g$

Solution

The constraint that $\theta_1 = \theta_3$ **yields the constraint equation**

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = T \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$ **where** $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ **. The new matrices are**

$$\overline{K} = T^T K T = \begin{bmatrix} 2(1+\gamma) & -\gamma \\ -\gamma & \frac{1}{2}(1+\gamma) \end{bmatrix}$$ **and**

$$\overline{M} = T^T M T = \begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

Solution

The eigen problem in this reduced system is

$$-\alpha^2\begin{bmatrix}2 & 0\\ 0 & 1/4\end{bmatrix}\begin{bmatrix}A_1\\ A_2\end{bmatrix}+\begin{bmatrix}2(1+\gamma) & -\gamma\\ -\gamma & \frac{1}{2}(1+\gamma)\end{bmatrix}\begin{bmatrix}A_1\\ A_2\end{bmatrix} = 0.$$

$$det\begin{bmatrix}2(1+\gamma)-2\alpha^2 & -\gamma\\ -\gamma & \frac{1}{2}(1+\gamma)-\frac{\alpha^2}{4}\end{bmatrix} = 0$$

This has a characteristic equation $\frac{\alpha^4}{2}-\frac{3}{2}(1+\gamma)\alpha^2+(1+2\gamma) = 0$

Solution

The solutions are $\alpha^2 = \frac{3}{2}(1+\gamma) \pm \frac{1}{2}\sqrt{1+2\gamma+9\gamma^2}$**. Substituting this back into the algebraic eigen value problem**

$$\begin{bmatrix} -(1+\gamma) \mp \sqrt{1+2\gamma+9\gamma^2} & -\gamma \\ -\gamma & \frac{1}{8}[(1+\gamma) \mp \sqrt{1+2\gamma+9\gamma^2}] \end{bmatrix} \begin{bmatrix} 1 \\ \frac{A_2}{A_1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

yields $\frac{A_2}{A_1} = -\frac{1+\gamma}{\gamma} \mp \frac{\sqrt{1+2\gamma+9\gamma^2}}{\gamma}$

Solution
Map these modes back to original space

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = T \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$ **where** $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$**, so our solutions are**

$$\left(\alpha^2 = \frac{3}{2}(1+\gamma) \pm \frac{1}{2}\sqrt{1+2\gamma+9\gamma^2}\right), \begin{bmatrix} 1 \\ -\frac{1+\gamma}{\gamma} \mp \frac{\sqrt{1+2\gamma+9\gamma^2}}{\gamma} \\ 1 \end{bmatrix}$$

Examine Solution

Consider case where gravity dominate over spring forces: $\gamma = 0$. The two solutions are $\alpha^2 = \frac{3}{2} \pm \frac{1}{2}$:

$\alpha^2 = 1$ **corresponds to** $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **and**

$\alpha^2 = 2$ **corresponds to** $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

as expected.

Examine Solution

Consider case where spring forces dominate over gravity: $\gamma \to \infty$. The two solutions are $\alpha^2 = \left(\frac{3}{2} \pm \frac{3}{2}\right)\gamma$:

$$\alpha^2 = 0 \quad \textbf{corresponds to} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ **(rigid body motion), and**

$$\alpha^2 = 3\gamma \textbf{ corresponds to} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

as expected.

Solutions to Governing Equations

Lets consider the solution $x(t)$ **to** $M\ddot{x}(t) + Kx(t) = F(t)$ **and expand** $x(t) = \sum_{k=1}^{N} \beta_k(t) x_k$ **where** x_k **is the k'th normalized eigenvector, corresponding to the k'th eigen value** ω_k^2**. This can be expressed in matrix form as** $x(t) = P\beta(t)$ **where** P **is modal matrix.**

This yields the system $\ddot{\beta} + D\beta = P^T F = \mathcal{F}$ **where** $D = P^T KP$ **is the diagonal matrix of eigen values.**

Solutions to Governing Equations

Lets solve for the β via Laplace Transforms:

$$(s^2 + \omega_k^2)\mathcal{L}\beta_k = s\beta_k(0) + \dot{\beta}_k(0) + \mathcal{L}(\mathcal{F}_k(t))(s)$$

This has the solution

$$\beta_k = \beta_k(0)\cos\omega_k t + \dot{\beta}_k(0)\sin\omega_k t + \int_0^t \mathcal{F}_k(t-\tau)\frac{\sin(\omega_k\tau)}{\omega_k}d\tau$$

where the vector $\dot{\beta}(0) = P^{-1}\dot{x}(0)$ **and** $\beta(0) = P^{-1}x(0)$

Solutions to Governing Equations With Damping

The above development provided the solutions for problem with no damping. The more general problem is where the damping is represented by a general matrix C. In general, this will yield complex modes, and we shall study that case later.

There is a class of damping matrices for which the elastic eigen modes are preserved. These problems are much more tractable. Because $P^T MP = I$ and $P^T KP = D$ we may conclude that $M = P^{-T} IP^{-1}$ and $K = P^{-T} DP^{-1}$.

This suggests looking for damping matrices $C = P^{-T} \chi P^{-1}$ where χ is also a diagonal matrix consisting of terms $2\zeta_k \omega_k$ on the diagonal.

Damping Matrices

We examine our dynamics equation

$$M\ddot{x} + C\dot{x} + Kx = F.$$

We diagonalize with the modal matrix to find

$$\ddot{\beta} + \chi\dot{\beta} + D\beta = P^T F = \mathcal{F}$$

Again, we solve for the β via Laplace Transforms:

$$(s^2 + 2\zeta_k\omega_k s + \omega_k^2)\mathcal{L}\beta_k = s\beta_k(0) + \dot{\beta}_k(0) + \mathcal{F}_k(t) \text{ for each } k.$$

Solution

Exploit that $\mathcal{L}^{-1}\left(\dfrac{1}{s^2 + 2\zeta_k\omega_k s + \omega_k^2}\right) = e^{-\zeta\omega t}\dfrac{\sin(\omega\sqrt{1-\zeta^2}t)}{\omega\sqrt{1-\zeta^2}}$

and that $\mathcal{L}^{-1}\left(\dfrac{s}{s^2 + 2\zeta_k\omega_k s + \omega_k^2}\right) = \dfrac{d}{dt}\left(e^{-\zeta\omega t}\dfrac{\sin(\omega\sqrt{1-\zeta^2}t)}{\omega\sqrt{1-\zeta^2}}\right)$

to solve for $\beta_k(t)$**.**

Transfer Functions
Synthesized from Modal Response

The whole system is represented in Laplace space by

$$\mathcal{L}(x) = P\mathcal{L}(\beta) = PG^dP^T\mathcal{L}(F)$$

where $G^d(s)$ is a diagonal matrix, whose diagonal terms are

$$G^d_{kk} = \frac{1}{s^2 + 2\zeta_k\omega_k s + \omega_k^2}$$

we have assumed homogeneous initial conditions.

Say we want to find the transfer function from force DOF n to displacement DOF m. Then

$$G_{mn}(s) = \sum_{k=1}^{\text{modes}} P_{mk}G^d_{kk}P_{nk}$$

Modal Damping

The assumption made above, that the damping matrix is diagonalizable, is the assumption of ***modal damping***.

Note that modal damping includes the special cases of proportional damping:

$$M\ddot{x} + (aM + bK)\dot{x} + Kx = F(t)$$

is diagonalized by the modal matrix. $x = P\beta$ as

$$\beta + (aI + bD)\dot{\beta} + D\beta = \mathcal{F} .$$

Modal Damping Terms

We see the advantages of assuming modal damping, the question becomes: “How do we deduce that damping ζ_k associated with each mode?”

There are three standard answers.

1. **Experimentally. Plot frequency response, identify peaks with eigen frequencies, and determine fraction for critical damping from the half power points.**

2. **Where a real damping matrix is known, compute $P^T CP$ and discard the off-diagonal terms**

3. **Where the damping mechanism can be modeled, set the model damping to reproduce the dissipation ratio associated with modal vibration. This is the modal strain energy method.**

Examples and discussion follow.

Next Time

More on Modal Damping and the Modal Strain Energy Method.

An Introduction to Linear Viscoelasticity.