Slides of Lecture 7

Today’s Class:

Conservation of Energy
Small Displacements and Linearization
Dissipation Potential

Mechanical Energy

Consider a system composed of N particles. For each of which,

$$m_k \frac{d}{dt}\vec{v}_k = \vec{F}_k.$$

We contract each side with the particle velocity and sum over particles:

$$\sum_k m_k\left(v_k \cdot \frac{dv_k}{dt}\right) = \sum_k \vec{F}_k \cdot \vec{v}_k$$

Note that $\sum_k m_k\left(v_k \cdot \frac{dv_k}{dt}\right) = \sum_k \frac{d}{dt}\left[\frac{m_k}{2}(v_k \cdot v_k)\right] = \frac{dT}{dt}$ **and that**

$\sum_k \vec{F}_k \cdot \vec{v}_k$ **is the rate at which work is applied to the system.**

Mechanical Energy

Always,

$$\frac{dT}{dt} = \frac{d}{dt}(\text{Work Done to System})$$

But for Conservative Systems,

$$\frac{dT}{dt} = \sum_k \vec{F}_k \cdot \vec{v}_k = \sum_k -\frac{\partial V}{\partial \vec{x}_k} \cdot \frac{d\vec{x}_k}{dt} = -\frac{dV}{dt}$$

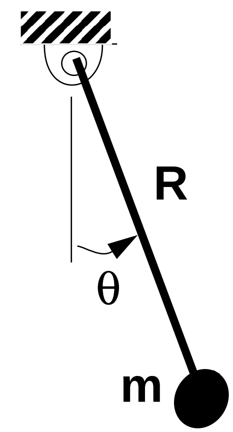
from which we conclude that

$$\frac{d}{dt}(T + V) = 0$$

Lets Apply Conservation of Energy To a Simple System.

A pendulum is released from rest at an angle of θ_0.

Calculate the maximum velocity achieved by the pendulum.



Initially $T_0 = 0$ **and** $V_0 = -mgR\cos\theta_0$**, so**

$$T_0 + V_0 = -mgR\cos\theta_0$$

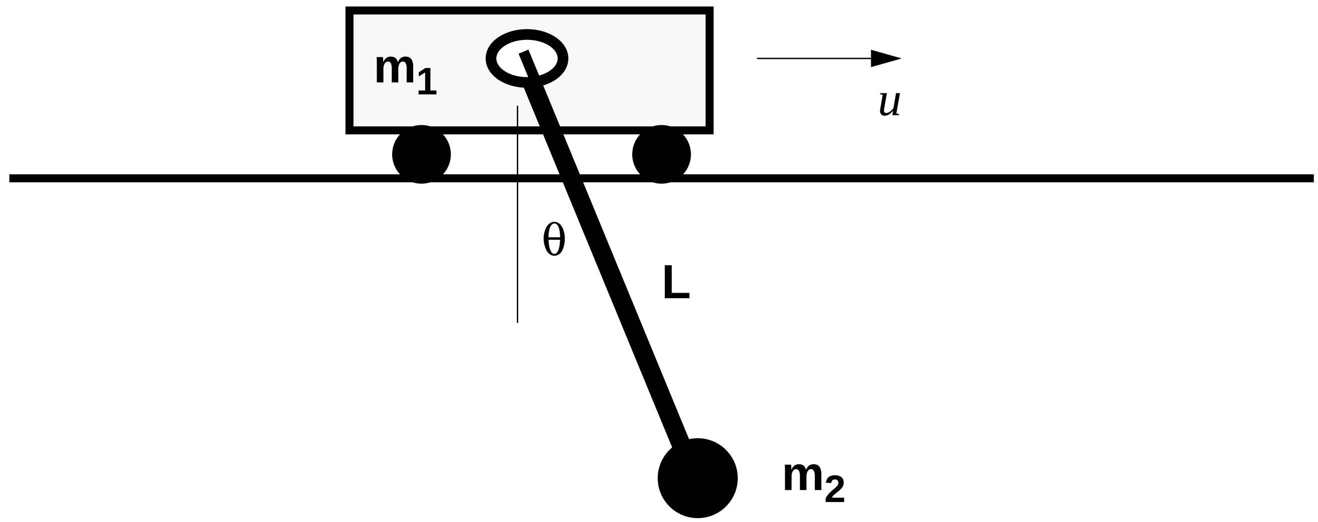
The potential energy is minimized when the pendulum is at the bottom of its swing. At that point, the kinetic energy must be at its maximum.

$$T|_{\theta=0} + V|_{\theta=0} = T_0 + V_0 \quad \Rightarrow \quad T|_{\theta=0} = mgR(1 - \cos\theta_0)$$

and $v_{max} = \sqrt{2gR(1 - \cos\theta_0)}$

Solution to Homework Assigned in Previous Lecture

Derive the equations of motion for the following 2-d.o.f. problem:



Express kinematics in terms of generalized degrees of freedom:

$\vec{x}_1 = u\vec{i}$ **and** $\vec{x}_2 = u\vec{i} + L\sin\theta\vec{i} - L\cos\theta\vec{j}$.

The potential energy is just that due to gravity: $V = -m_2 g L\cos\theta$

The generalized forces are:

$$F_u = -\frac{\partial V}{\partial u} = 0 \text{ and } F_\theta = -\frac{\partial V}{\partial \theta} = -m_2 g L \sin\theta .$$

The kinetic energy is

$$T = \frac{1}{2} m_1 \dot{\vec{x}}_1 \cdot \dot{\vec{x}}_1 + \frac{1}{2} m_2 \dot{\vec{x}}_2 \cdot \dot{\vec{x}}_2$$

$$= \frac{1}{2} m_1 \dot{u}^2 + \frac{1}{2} m_2 (\dot{u} + L\dot{\theta}\cos\theta)^2 + \frac{1}{2} m_2 (L\dot{\theta}\sin\theta)^2$$

$$= \frac{1}{2}(m_1 + m_2)\dot{u}^2 + m_2 L\dot{\theta}\dot{u}\cos\theta + \frac{1}{2} m_2 L^2 \dot{\theta}^2$$

Lets calculate the appropriate partial derivative from this

$\frac{\partial T}{\partial u} = 0$, **and**

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = \frac{d}{dt}[(m_1 + m_2)\dot{u} + m_2 L\dot{\theta}\cos\theta]$$

$$= (m_1 + m_2)\ddot{u} - m_2 L\dot{\theta}^2\sin\theta + m_2 L\ddot{\theta}\cos\theta$$

$\frac{\partial T}{\partial \theta} = -m_2 L\dot{\theta}\dot{u}\sin\theta$ **and**

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}[m_2(L\dot{u}\cos\theta + L^2\dot{\theta})]$$

$$= m_2 L(\ddot{u}\cos\theta + L\ddot{\theta} - \dot{u}\dot{\theta}\sin\theta)$$

Adding all the Ingredients

$$(m_1 + m_2)\ddot{u} - m_2 L\dot{\theta}^2 \sin\theta + m_2 L\ddot{\theta}\cos\theta = 0$$

and $m_2 L\ddot{u}\cos\theta + m_2 L^2\ddot{\theta} = -m_2 g L\sin\theta$

These are two messy equations to solve.

Lets Simplify

$$(m_1 + m_2)\ddot{u} - m_2 L \underbrace{\dot{\theta}^2}_{0} \sin\theta + m_2 L \ddot{\theta} \cos\theta = 0$$

and $m_2 L \ddot{u} \cos\theta + m_2 L^2 \ddot{\theta} = -m_2 g L \sin\theta$

Lets introduce the assumption that θ and $\dot{\theta}$ are each so small that we can ignore powers and products of them. From this we get the small-angle formulae:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots \approx \theta \text{ and } \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \ldots \approx 1$$

We now have

$(m_1 + m_2)\ddot{u} - (m_2 L\ddot{\theta}) = 0$ **and**

$m_2 L\ddot{u} + m_2 L^2\ddot{\theta} = -m_2 gL\theta$**.**

In matrix form, this is

$$\begin{bmatrix} m_1 + m_2 & m_2 L \\ m_2 L & m_2 L^2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{\theta} \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & m_2 gL \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which is our canonical form for linear the undamped vibration problem

Steady State Solution

Lets see what kinds of solution we might get if we postulate steady-state solutions of the form $\begin{bmatrix} u \\ \theta \end{bmatrix} = Re\left\{ \begin{bmatrix} \hat{u} \\ \hat{\theta} \end{bmatrix} e^{i\omega t} \right\}$**. Substituting this into the matrix equation, we find**

$$Re\left\{ \begin{bmatrix} -\omega^2(m_1 + m_2) & -\omega^2 m_2 L \\ -\omega^2 m_2 L & -\omega^2 m_2 L^2 + m_2 g L \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{\theta} \end{bmatrix} e^{i\omega t} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

from which we deduce that

$$\begin{bmatrix} -\omega^2(m_1 + m_2) & -\omega^2 m_2 L \\ -\omega^2 m_2 L & -\omega^2 m_2 L^2 + m_2 g L \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Steady State Solution

The equation $\begin{bmatrix} -\omega^2(m_1 + m_2) & -\omega^2 m_2 L \\ -\omega^2 m_2 L & -\omega^2 m_2 L^2 + m_2 g L \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **has a non-trivial solution only if**

$$det\left\{ \begin{bmatrix} -\omega^2(m_1 + m_2) & -\omega^2 m_2 L \\ -\omega^2 m_2 L & -\omega^2 m_2 L^2 + m_2 g L \end{bmatrix} \right\} = 0$$

$$m_2 \omega^2 L [m_1 L \omega^2 + (m_1 + m_2) g] = 0$$

which has two solutions.

Solutions

- **Solution 1:** $\omega^2 = 0$**. Substituting this into the matrix equation we see** $\begin{bmatrix} \hat{u} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$**. This is translation at constant speed.**

- **Solution 2.** $\omega^2 = -\dfrac{(m_1 + m_2)g}{m_1 L}$**. Substituting this into the matrix equation, we find** $\begin{bmatrix} \hat{u} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ -\dfrac{(m_1 + m_2)g}{m_2 L} \end{bmatrix}$**. Lets discuss the significance of this.**

Linearization of Vibration Problems. Perturbation about Static Equilibrium

Say we have derived our non-linear vibration equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = \sum_k \vec{F}_k^e(t) \cdot \frac{\partial \vec{x}_k}{\partial q_r} = F_r^e(\{q\}, t).$$

We linearize in the following steps:

1. **Solve for a stable equilibrium configuration** $\{q^s\}$**. The system will be in equilibium if** $\left.\frac{\partial V}{\partial q_r}\right|_{\{q\} = \{q^s\}} = 0$ **for each** r **and will be in stable equilibrium if** V **is at a local minumum.**

Perturbation about Static Equilibrium

2. **express $\{q\} = \{q^s\} + \{\Delta q\}$, and substitute into the governing equation. In this substitution, we use the Taylor series expansion:**

$$f(\{q^s\} + \{\Delta q\}) = f(\{q^s\}) + \sum_r \left(\frac{\partial f}{\partial q_r}\right)\bigg|_{\{q^s\}} \Delta q_r$$

3. **delete all products and powers of Δq_r and $\Delta \dot{q}_r$.**

4. **group terms to form the mass and stiffness matrix.**

Lets try this out on some simple problems

Consider the problem of an inverted pendulum. $T = \frac{m}{2}R^2\dot{\theta}^2$,

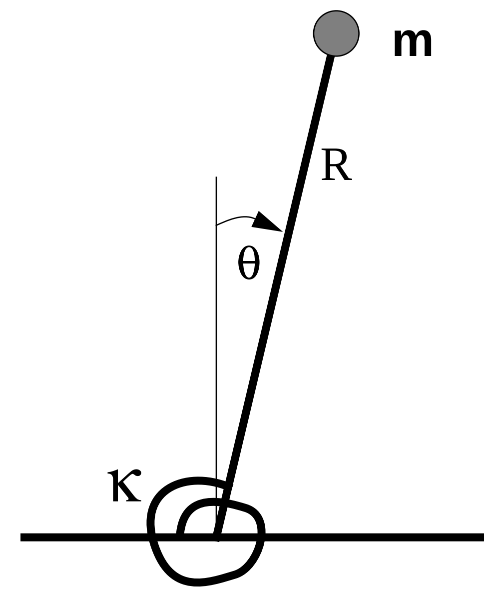
$V = \frac{\kappa}{2}\theta^2 + mgR\cos\theta$, yielding

$$mR^2\ddot{\theta} + \kappa\theta - mgR\sin\theta = 0.$$

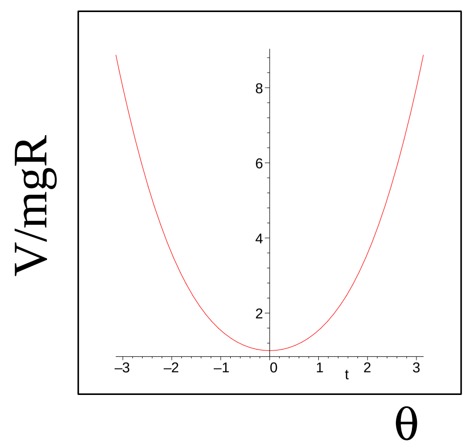
(We have done this before),

Say $\kappa < mgR$, then $\theta = 0$ is no longer a configuration of static equilibrium. (How do we know? The negative stiffness is a clue.)

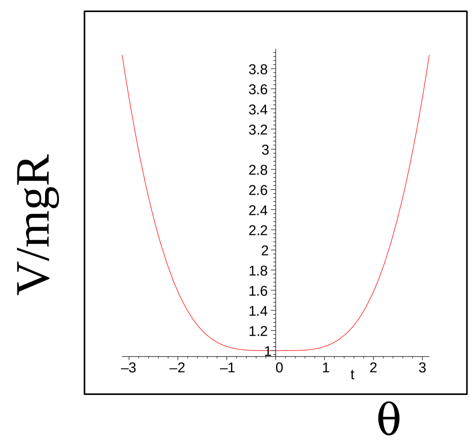
$$\vec{x} = R(\sin\theta\vec{i} + \cos\theta\vec{j})$$



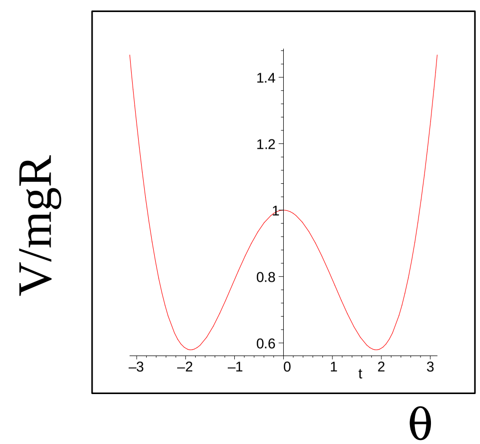
Examine Potential Energy for different κ / mgR



κ=2mgR



κ=mgR



κ=mgR/2

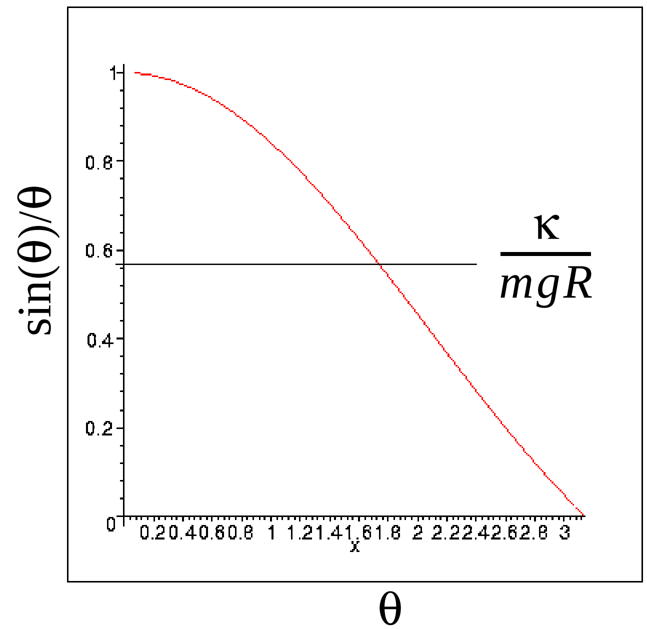
We see that the location $\theta = 0$ is not a *stable* equilibrium for $\kappa < mgR$

Linearization

1. **We now solve for static equilibrium:** $\kappa\theta^s - mgR\sin\theta^s = 0$

   This may be done numerically:

Solving $\dfrac{\sin\theta^s}{\theta^s} = \dfrac{\kappa}{mgR}$



From the above plot we may pick off the value of θ^s that satisfies the equilibrium equation.

Formulation of Linearized Equations

2. **substitute** $\{q\} = \{q^s\} + \{\Delta q\}$ **into governing equation** $mR^2\Delta\ddot{\theta} + \kappa(\theta^s + \Delta\theta) - mgR\sin(\theta^s + \Delta\theta) = 0$**, and expand with Taylor series**

$$mR^2\Delta\ddot{\theta} + \kappa(\theta^s + \Delta\theta) - mgR(\sin(\theta^s) + \cos(\theta^s)\Delta\theta)$$
$$= mR^2\Delta\ddot{\theta} + \underbrace{[\kappa\theta^s - mgR\sin(\theta^s)]}_{0} + \kappa\Delta\theta - mgR(\cos(\theta^s)\Delta\theta)$$
$$= mR^2\Delta\ddot{\theta} + (\kappa - mgR\cos(\theta^s))\Delta\theta = 0$$

Note that the equation for static equilibrium appears and is identically zero.

3. **delete all products and powers of** Δq_r **and** $\Delta\dot{q}_r$**. Done.**

4. **group terms to form the mass and stiffness matrix. Done.**

The Compound Pendulum Problem
Lets derive the governing equations

Kinematics: $\vec{x}_1 = R_1(\sin\theta_1\vec{i} - \cos\theta_1\vec{j})$.

$$x_2 = R_1(\sin\theta_1\vec{i} - \cos\theta_1\vec{j})$$

and

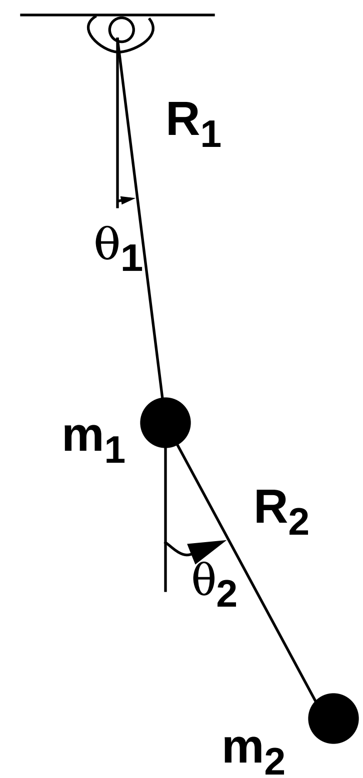
$$+ R_2(\sin\theta_2\vec{i} - \cos\theta_2\vec{j}).$$

Velocities: $\dot{\vec{x}}_1 = R_1\dot{\theta}_1(\cos\theta_1\vec{i} + \sin\theta_1\vec{j})$

$$\dot{\vec{x}}_2 = R_1\dot{\theta}_1(\cos\theta_1\vec{i} + \sin\theta_1\vec{j})$$

and

$$+ R_2\dot{\theta}_2(\cos\theta_2\vec{i} + \sin\theta_2\vec{j})$$



Compound Pendulum

Kinetic Energy:

$$T = \frac{m_1}{2}(R_1\dot{\theta}_1)^2 + \frac{m_2}{2}[(R_1\dot{\theta}_1)^2 + (R_2\dot{\theta}_2)^2] + m_2(R_1\dot{\theta}_1)(R_2\dot{\theta}_2)\cos(\theta_2 - \theta_1)$$

Lets do the exercise:

$$\frac{\partial T}{\partial \dot{\theta}_1} = (m_1 + m_2)R_1^2\dot{\theta}_1 + m_2 R_1 R_2 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = (m_1 + m_2)R_1^2\ddot{\theta}_1 + m_2 R_1 R_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) + m_2 R_1 R_2 \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_2 - \theta_1) - m_2 R_1 R_2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1)$$

$$\frac{\partial T}{\partial \theta_1} = m_2 R_1 R_2 \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_2 - \theta_1)$$

Compound Pendulum

$$\frac{\partial T}{\partial \dot{\theta}_2} = m_2 R_2^2 \dot{\theta}_2 + m_2 R_1 R_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = m_2 R_2^2 \ddot{\theta}_2 + m_2 R_1 R_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1)$$
$$- m_2 R_1 R_2 \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_2 - \theta_1) + m_2 R_1 R_2 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1)$$

$$\frac{\partial T}{\partial \theta_2} = -m_2 R_1 R_2 \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_2 - \theta_1)$$

Compound Pendulum

Potential Energy:

$$V = m_1 g y_1 + m_2 g y_2 =$$

$$= -m_1 g R_1 \cos\theta_1 - m_2 g [R_1 \cos\theta_1 + R_2 \cos\theta_2]$$

$\frac{\partial V}{\partial \theta_1} = g(m_1 + m_2) R_1 \sin\theta_1$ **and** $\frac{\partial V}{\partial \theta_2} = g m_2 R_2 \sin\theta_2$

Put them all together:

$$(m_1 + m_2) R_1^2 \ddot{\theta}_1 + m_2 R_1 R_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) - m_2 R_1 R_2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) + g(m_1 + m_2) R_1 \sin\theta_1 = 0$$

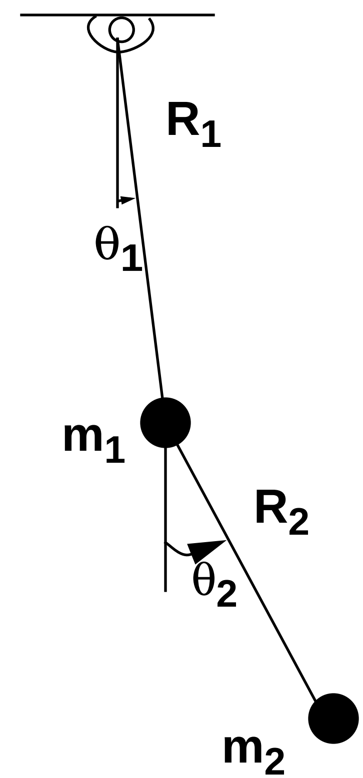
and

$$m_2 R_2^2 \ddot{\theta}_2 + m_2 R_1 R_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) + m_2 R_1 R_2 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) + g m_2 R_2 \sin\theta_2 = 0$$

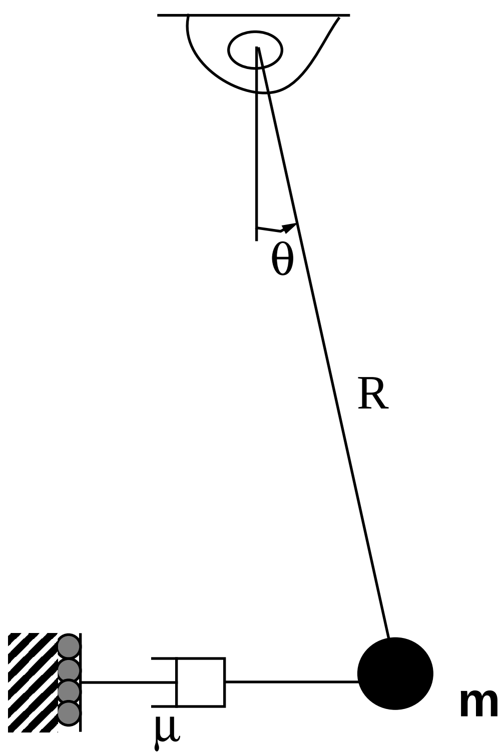
Homework

1. **The above derivation was done on the fly. Please check that it is correct.**

2. **Using the methods discussed above, identify the equilibrium configuration and linearize the equations of motion about that configuration.**



Rayleigh Damping



For the special case of viscous damping, there exists a dissipation potential that facilitates the formulation of the equations of motion. Lets look at the example shown.

Recall that $\vec{x} = R(\sin\theta\vec{i} - \cos\theta\vec{j})$ **The restraining force on the pendulum is found to be** $\overrightarrow{F^D} = -\mu\dot{x}_1\vec{i} = -\mu R\dot{\theta}\cos\theta\vec{i}$**. The corresponding generalized force associated with the** θ **direction is**

$$F^D{}_\theta = \overrightarrow{F^D} \cdot \frac{\partial}{\partial\theta}\vec{x} = -\mu R^2\dot{\theta}(\cos\theta)^2$$

The governing equation is found to be

$$mR^2\ddot{\theta} + Rmg\sin\theta = -\mu(R\cos\theta)^2\dot{\theta}$$

Rayleigh Dissipation Potential

There is another way to calculate the generalized force associated with the dashpot.

The rate of dissipation in the dashpot is $D = \overrightarrow{F^D} \cdot \dot{\vec{x}} = \mu \dot{x}_1^2$**. We now evaluate**

$$\tilde{F}^D_\theta = -\frac{\partial}{\partial \dot{\theta}}(D/2) = -\frac{\partial}{\partial \dot{\theta}}\left(\frac{\mu}{2}(R\cos\theta)^2\dot{\theta}^2\right)$$

$$= -\mu(R\cos\theta)^2\dot{\theta}$$

Notice that $\tilde{F}^D_\theta = F^D_\theta$**.**

Lets discuss why this should not surprise us. Consider virtual work.

Rayleigh Dissipation Function

The Rayleigh dissipation function provides a convenient method for calculating the generalized force associated with dashpots.

The recipe is as follows

1. calculate the rate of dissipation, D, in terms of the speed in cartesian coordinates.

2. convert that expression to one involving generalized coordinates and generalized speeds.

3. make the partial with respect to each generalized speed and multiply by $-(1/2)$

Rayleigh Dissipation Function Warning

Though the Rayleigh dissipation function is easy to use, its uses are limited.

- **Only in a minority of circumstances is the damping associated with discrete dashpots.**
- **Usually, damping is associated with a material dissipation. We will discuss this later if time permits.**