

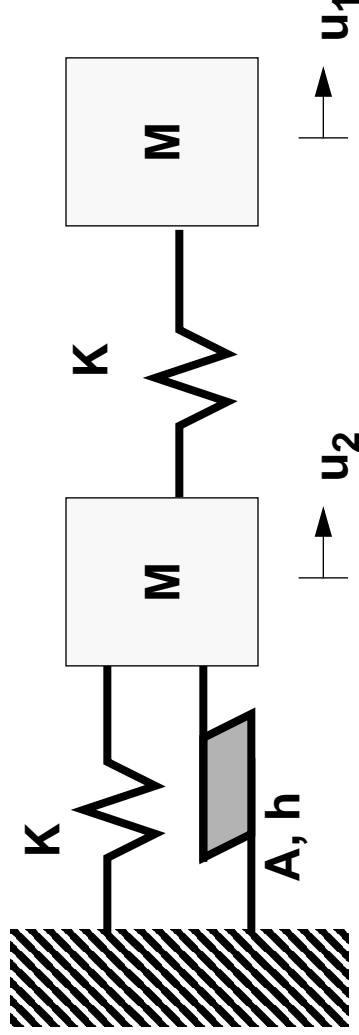
Take Home Final Exam A Damping Problem

Advanced Vibrations

You may use any references, but do not ask anyone for help.

Please put you finished exams in an envelope and place that in my mail slot no later than noon Wednesday 16 December 1998.

We consider a structure such as that shown below



We have two masses, two springs, and a shear damper. The cross sectional area of the damper is A and the thickness of the damping material is h . The damping material is that discussed earlier. For

$M = 300 \text{ kg}$; $K = 1.0 \times 10^8 \text{ N/m}$; $A = 0.01 \text{ m}^2$, and $h = 0.01 \text{ m}$

Problem

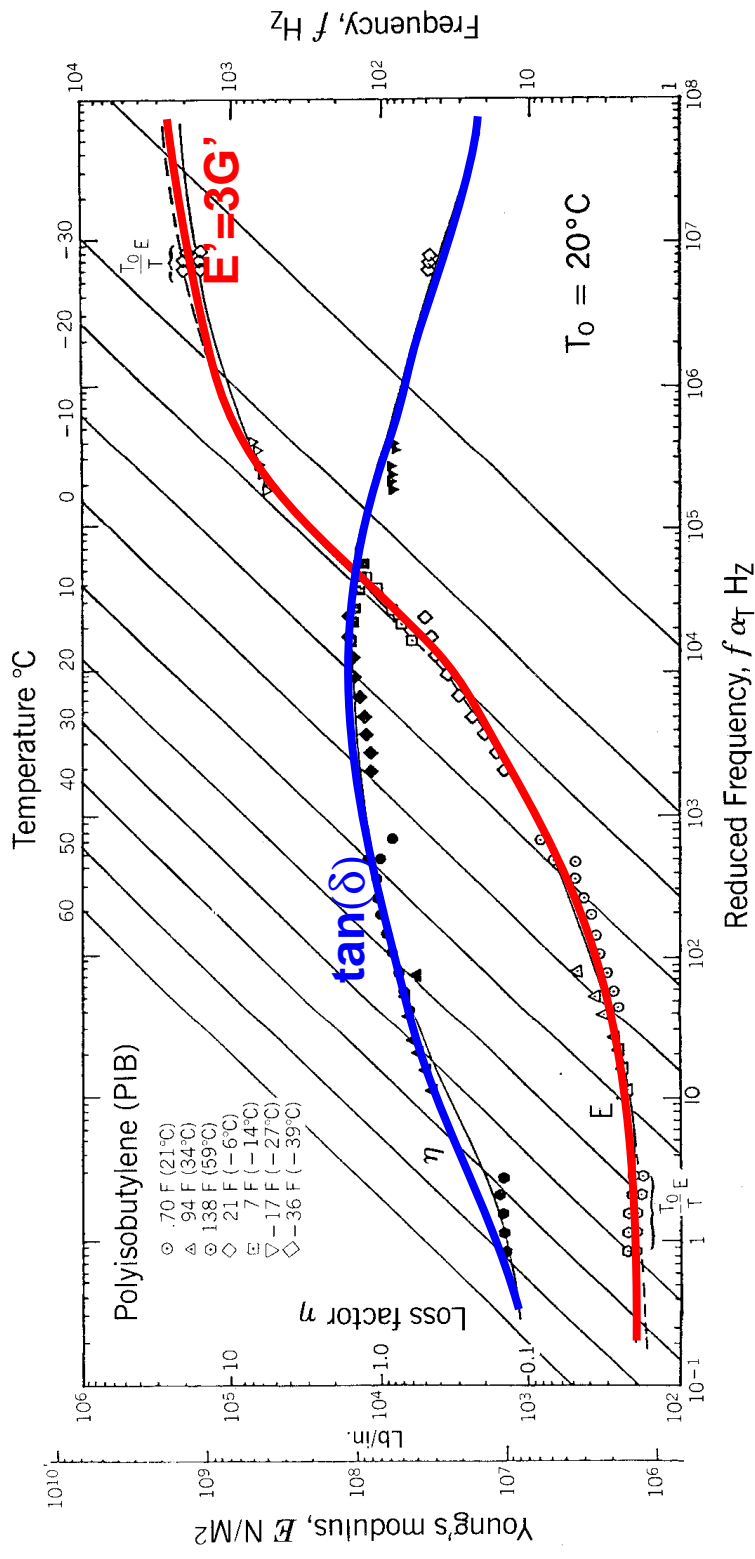
Use Matlab in any parts of the following that you wish

1. Solving the governing equation in frequency space, plot the magnitude of the driving point frequency response function versus frequency.
2. From the above, calculate the fraction of critical damping for each mode using half-power points.
3. Find the elastic eigenmodes and frequencies. You may ignore the slider for this, but you must tell me why.
4. Using the modal strain energy method, calculate the fraction of critical damping for each mode. These should be similar to the values calculated above.
5. Assuming modal damping, solve the problem for the following initial conditions: $u_1(0) = 0, u_2(0) = 0, \dot{u}_1(0) = 1, u_2(0) = 0$, and plot $\dot{u}_1(t)$ and $\dot{u}_2(t)$ for several cycles of the lower frequency.

The Damping Material

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The following plot for polyisobutylene is taken from “Vibration Damping” by Nashif et. al.



011C. Nomogram.

Interpolating Functions

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We can fit the above plots reasonably well in $0.1 < \frac{\omega}{2\pi} < 10^5$ by

$$\text{Log}(3G'(\omega)) = \text{Log}(E'(\omega)) = 6.2 + 0.0112 \left(\text{Log} \left(\frac{\omega}{2\pi} \right) \right)^{3.3} \quad \text{or}$$

$$5.72 + 0.0112 \left(\text{Log} \left(\frac{\omega}{2\pi} \right) \right)^{3.3}$$

$$G'(\omega) = 10$$

and

$$\text{Log}(\tan \delta(\omega)) = -1 + 0.56 \text{Log} \left(\frac{\omega}{2\pi} \right) - 0.069 \left(\text{Log} \left(\frac{\omega}{2\pi} \right) \right)^2 \quad \text{so}$$

$$G''(\omega) = G'(\omega) \tan \delta(\omega)$$

$$4.72 + 0.56 \text{Log} \left(\frac{\omega}{2\pi} \right) - 0.069 \left(\text{Log} \left(\frac{\omega}{2\pi} \right) \right)^2 + 0.0112 \left(\text{Log} \left(\frac{\omega}{2\pi} \right) \right)^{3.3} \\ = 10$$

Solution Part 1

1. Solving the governing equation in frequency space, plot the magnitude of the driving point frequency response function versus frequency.
2. From the above, calculate the fraction of critical damping for each mode using half-power points.

In frequency space, the governing equation is

$$-\omega^2 \begin{bmatrix} M & \\ & M \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & 2k + G^*(\omega) \frac{A}{h} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix}$$

We write this as $\begin{bmatrix} A(\omega) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix}$. The frequency response is found by inverting $\begin{bmatrix} A(\omega) \end{bmatrix}$.

Solution: Part 1

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = [A(\omega)]^{-1} \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix} = H(\omega) \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix}$$

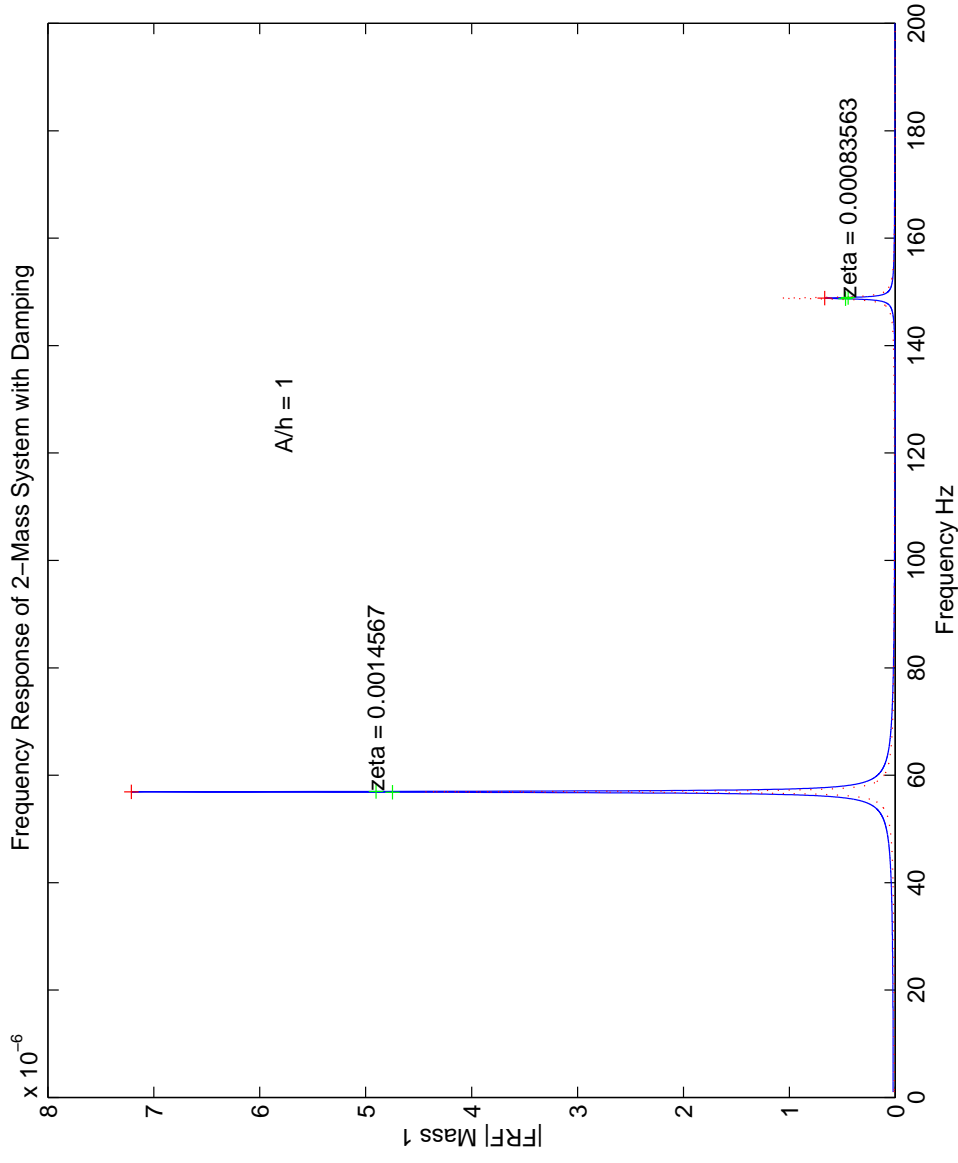
Say we apply we apply the driving force to mass 1, then the driving point frequency response is $H_{11}(\omega)$. If we drive mass 2, the driving point frequency response is $H_{22}(\omega)$.

Both driving point frequency response functions are plotted in the following figure. The fractions of critical damping are calculated from $H_{11}(\omega)$, but the damping numbers are identical when computed from $H_{22}(\omega)$.

A very fine discretization in frequency space is necessary to converge on the fractions of critical damping. Thirty intervals/Hz were used to calculate the following.

Solution: Part 2

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Computed damping is 0.15% and 0.084% for the first and second mode, respectively.

Solution Part 2

The reason why computed damping does not depend on which mass is driven becomes clear when we examine the modal expression for nodal displacement:

$$X(\omega) = \sum_k \frac{P_k(P_k^T F(\omega))}{(\omega_k^2 - \omega^2) + 2i\zeta_k \omega \omega_k}$$

When the system is driven at frequency ω , only those modes with frequency in that vicinity are excited. We see that all of the physical nodes will move proportionally in response to that excitation. Hence the half-power calculation for damping will be independent of which node is excited.

Solution: Part 3

Compute elastic eigen modes and frequencies.

From the analysis of part 1, the eigen problem is

$$-\omega^2 \begin{bmatrix} M \\ M \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & 2k + G^*(\omega) \frac{A}{h} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In the frequency range of interest,

$$\frac{A}{h} G''(\omega) < \frac{A}{h} G'(\omega) < 8.0 \times 10^5 N/m \ll k = 1.0 \times 10^8 N/m$$

and it is not anticipated that the damper will significantly affect the natural frequencies or modes.

Solution: Part 3

The eigen problem can be made non-dimensional

$$\begin{bmatrix} 1 - \alpha^2 & -1 \\ -1 & 2 - \alpha^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{where } \alpha^2 = \omega^2 \left(\frac{M}{K} \right).$$

The eigen values are $\alpha^2 = \left(\frac{3 - \sqrt{5}}{2}, \frac{3 + \sqrt{5}}{2} \right) = (0.382, 2.62)$

In terms of angular frequency,

$$\omega^2 = \alpha^2 \left(\frac{K}{M} \right) = (1.27 \times 10^5, 8.73 \times 10^5).$$

The natural frequencies are $(\omega_1, \omega_2) = (357, 934)$ and

$$(f_1, f_2) = \frac{1}{2\pi}(\omega_1, \omega_2) = (56.8, 149) \text{ Hz}$$

Solution: Part 3

The corresponding eigenvectors are

$$\begin{pmatrix} \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{\sqrt{5}+1}{2} \end{bmatrix} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} 1 \\ 0.618 \end{bmatrix}, \begin{bmatrix} 1 \\ -1.62 \end{bmatrix} \end{pmatrix}.$$

These have not been normalized. They do not have to be for this application, but we shall normalize them later on.

Solution: Part 4

Modal Strain Energy for First Mode

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The first mode looks like $\begin{bmatrix} 1 \\ 0.618 \end{bmatrix}$ and occurs at a

$$\omega_1 = 357, f_1 = 56.8 \text{ Hz}.$$

The maximum strain energy in a cycle of this mode is

$$E_S = \frac{k}{2}(1 - 0.618)^2 + \frac{k}{2}(0.618)^2 = 26.4 \times 10^6 \text{ NM}$$

The maximum energy dissipated in a cycle in of this mode is

$$E_D = \pi \frac{A}{h} G''(\omega)(0.618)^2 = \pi 0.365 \times 10^6 (0.618)^2 = 0.438 \times 10^6$$

$$\zeta = \frac{\eta}{2} = \frac{E_D}{4\pi E_S} = 0.0013$$

Solution: Part 4

Modal Strain Energy for Second Mode

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The first mode looks like $\begin{bmatrix} 1 \\ -1.62 \end{bmatrix}$ and occurs at a

$$\omega_2 = 934, f_2 = 149 \text{ Hz}.$$

The maximum strain energy in a cycle of this mode is

$$E_S = \frac{k}{2}(1 + 1.62)^2 + \frac{k}{2}(1.62)^2 = 4.74 \times 10^8 \text{ Nm}$$

The maximum energy dissipated in a cycle in of this mode is

$$E_D = \pi \frac{A}{h} G''(\omega)(1.62)^2 = \pi(1)5.7 \times 10^5 (1.62)^2 = 4.7 \times 10^6 \text{ Nm}$$

$$\zeta = \frac{\eta}{2} = \frac{E_D}{4\pi E_S} = 0.00079$$

Solution: Part 5

Solution with initial conditions

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From this point on, we shall need to normalize the eigen modes.

Recall that the physical degrees of freedom and the modal coordinates

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = P \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \end{bmatrix} \quad \text{where } P^T M P = I \text{ and}$$

$$P^T K P = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}.$$

Normalizing the eigen solutions calculated earlier,

$$P = \begin{bmatrix} 4.91 & -3.04 \\ 3.04 & 4.91 \end{bmatrix} \times 10^{-2}$$

Solution: Part 5

Solution with initial conditions

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There are no forcing terms, so the modal equations are of the form

$$\ddot{\beta}_n + 2\zeta\omega_n\dot{\beta}_n + \omega_n^2\beta_n = 0.$$

Since this is an “under damped” system, the solution is

$$\beta_n(t) = e^{-\zeta\omega_n t} [C_n \cos(t\omega_n \sqrt{1 - \zeta_n^2}) + D_n \sin(t\omega_n \sqrt{1 - \zeta_n^2})].$$

The time derivative of β_n is

$$\begin{aligned}\dot{\beta}_n(t) = & -\zeta_n\omega_n e^{-\zeta_n\omega_n t} [C_n \cos(t\omega_n \sqrt{1 - \zeta_n^2}) + D_n \sin(t\omega_n \sqrt{1 - \zeta_n^2})] \\ & + \omega_n \sqrt{1 - \zeta_n^2} (e^{-\zeta_n\omega_n t} [D_n \cos(t\omega_n \sqrt{1 - \zeta_n^2}) - C_n \sin(t\omega_n \sqrt{1 - \zeta_n^2})])\end{aligned}$$

Solution: Part 5

Solution with initial conditions

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Substituting into our equation for physical displacements,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.0491 \\ 0.0304 \end{bmatrix} \beta_1(t) + \begin{bmatrix} -0.0304 \\ 0.0491 \end{bmatrix} \beta_2(t)$$

Our initial conditions are $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In terms of our modal coordinates,

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.0491 \\ 0.0304 \end{bmatrix} C_1(0) + \begin{bmatrix} -0.0304 \\ 0.0491 \end{bmatrix} C_2(0)$$

and we conclude that $C_1 = C_2 = 0$.

Solution: Part 5

Solution with initial conditions

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$$\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0.0491 \\ 0.0304 \end{bmatrix} \omega_1 \sqrt{1 - \zeta_1^2} D_1 \\ + \begin{bmatrix} -0.0304 \\ 0.0491 \end{bmatrix} \omega_2 \sqrt{1 - \zeta_2^2} D_2 \\ = \begin{bmatrix} 17.6 \\ 10.9 \end{bmatrix} D_1 + \begin{bmatrix} -28.4 \\ 46 \end{bmatrix} D_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which is solved $D_1 = 0.041, D_2 = -0.010$

Solution: Part 5

Solution with initial conditions

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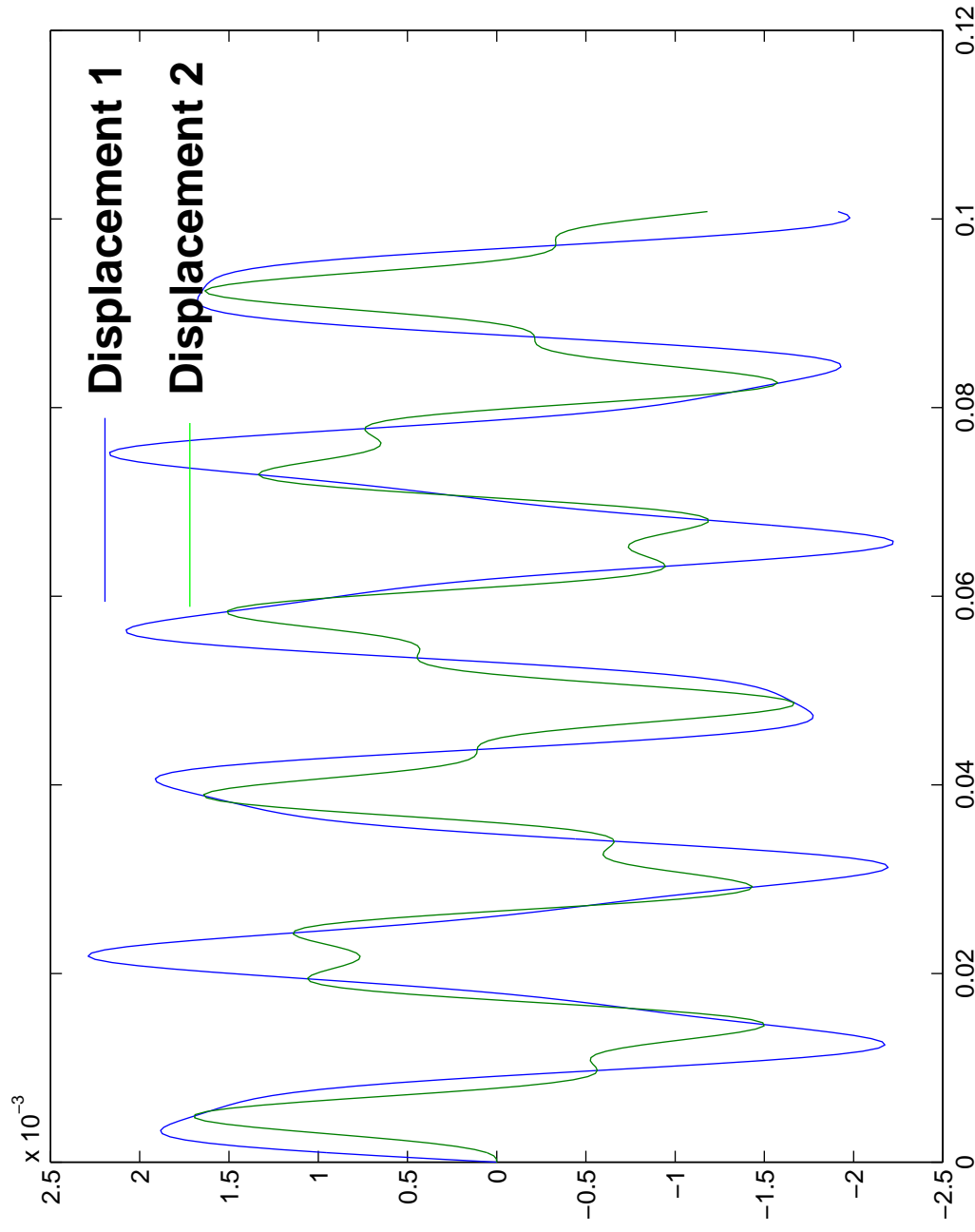
The calculated displacement is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.00202 \\ 0.00125 \end{bmatrix} e^{-0.52t} \sin(t358) + \begin{bmatrix} 0.00030 \\ -0.00048 \end{bmatrix} e^{-0.78t} \sin(t935)$$

Solution: Part 5

Computed Displacements

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Solution: Part 5 Computed Velocities

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