Slides of Lecture 11

Today's Class:

Eigen Analysis

This topic brings together our experience with matrix transformations and with integrating systems of equations to motivate the search and use of generalized eigen solutions.

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Integration of Linearized Equations

Say that we have derived governing equations for a vibrating system

$$M\ddot{x} + Kx = F$$

To integrate this numerically, we might pose it in a state space formalism:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \dot{\begin{bmatrix} \dot{x} \\ x \end{bmatrix}} = -\begin{bmatrix} & K \\ -K & \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

This can be rearranged to $\dot{y} = -Ay + \tilde{F}$

where $A = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}^{-1} \begin{bmatrix} & K \\ -K & \end{bmatrix}$ **and** $\tilde{F} = \begin{bmatrix} F \\ 0 \end{bmatrix}$.

We have integrated these equations numerically in the past without a lot of difficulty, but this approach can be impractical for large problems.

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Integration of Linearized Equations

For large problems, integrating $\dot{y} = -Ay + \tilde{F}$, which involves inverting the mass matrix once, but performing large matrix-vector multiplies at every time-step can be daunting. We would like to reduce the size of the algebraic problem that must be solved at each time step. Modal analysis lets us do that.

Further, modal analysis provides additional advantages:

- **providing a compact representation of the system dynamics**
- **facilitating modal truncation for model reduction**
- **providing a simple route to calculating transfer functions**

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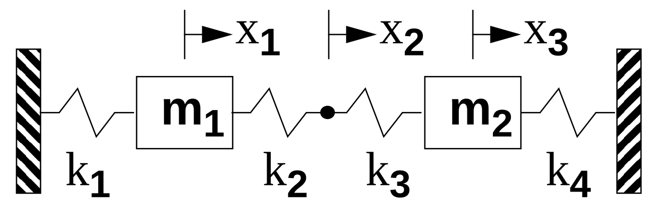
A Digression Why Do We Not Worry About The Singularity Of M?

Usually the mass matrix is constructed to be nonsingular, but it can happen that one introduces massless degrees of freedom. Consider the following structure. It has two masses, but three degrees of freedom. The node connecting the two springs is massless.



The kinetic energy is $T = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2$. The mass matrix is

$$M_{ij} = \frac{\partial^2 T}{\partial x_i \partial x_j}$$ so $$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_2 \end{bmatrix}$$, which is singular. We

demonstrate singularity by observing that $a^T M a = 0$ when

$$a^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

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Why Do We Not REALLY Worry About The Singularity Of M?

We would really like to condense the degree of freedom x_2 out of the problem. With the aid of the potential energy, we obtain an equation for x_2 in terms of x_1 and x_3.

$$V = \frac{k_1}{2}x_1^2 + \frac{k_2}{2}(x_2 - x_1)^2 + \frac{k_3}{2}(x_3 - x_2)^2 + \frac{k_4}{2}x_3^2.$$

Equilibrium of the node at x_2 requires that

$$0 = \frac{\partial V}{\partial x_2} = -k_1 x_1 + (k_1 + k_2)x_2 - k_3 x_3$$

from which we can solve for x_2: $x_2 = \frac{k_1}{k_1 + k_2}x_1 + \frac{k_3}{k_2 + k_3}x_3$.

(Compare the above to the second row of the stiffness matrix.)

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Why Do We Not REALLY Worry About The Singularity Of M?

We use the above constraint to reduce the size of our system:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \qquad \text{where } T = \begin{bmatrix} 1 & 0 \\ \dfrac{k_1}{k_1 + k_2} & \dfrac{k_3}{k_2 + k_3} \\ 0 & 1 \end{bmatrix}.$$

Our new governing equations are $T^T M T \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_3 \end{bmatrix} + T^T K T \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = T^T F$

where M, K, and F are the mass matrix, stiffness matrix, and applied force of the original problem.

In the rest of this development, we shall assume that the mass matrix is positive definite.

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Eigen Analysis
Finding a de-coupled model for the system

Now that we know a lot about the effects of performing a linear transformation of our variables, we can pose the following question:

Given a system defined by mass matrix M, mass matrix K, and displacement variables x, does there exist a transformation matrix T a new set of variables $x = T\beta$ and such that $T^T M T$ and $T^T K T$ are each diagonal?

The answer will be YES, but we shall have to prove it by demonstrating the transformation matrix that does the trick. We shall find that this matrix is one whose columns are the eigenvectors of the M, K system.

An eigen-pair (eigen solution) of the generalized eigen-problem of M and K are an eigenvalue λ and an eigenvector x such that

$$-\lambda M x + K x = 0$$

An eigen-pair is often represented as (λ, x).

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Eigen Analysis
Does a solution exist?

How do we know that even one eigen-solution exists? Consider the following extreme value problem.

From among all column vectors X that are not identically zero, find the column vector X_1 such that $R_1(X_1) = \dfrac{X_1^T K X_1}{X_1^T M X_1}$ is at a minimum.

Let $\lambda_1 = R_1(X_1)$ then for any y such that $y^T M y > 0$,

$$\frac{d}{d\alpha}\left[\frac{(X_1 + \alpha y)^T K (X_1 + \alpha y)}{(X_1 + \alpha y)^T M (X_1 + \alpha y)}\right]\Bigg|_{\alpha = 0} = 0.$$

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Eigen Analysis
Does a solution exist?

After a bunch of linear analysis, and exploiting the symmetry of K and M, we find that the above condition requires that $y^T(KX_1 - \lambda_1 MX_1) = 0$ for all nonzero y. From which we conclude that $KX_1 - \lambda_1 MX_1 = 0$. This proves the existence of the lowest eigenvalue.

The existence of the next higher eigenvalue is proven by considering a slightly modified minimization: Let

$$\lambda_2 = R_2(X_2) = \min_{\substack{0 < X^T MX \\ X_1^T MX = 0}} \frac{X^T KX}{X^T MX}$$

. An evaluation of the appropriate derivative will show that $KX_2 - \lambda_2 MX_2 = 0$. In this manner, we demonstrate the existence of as many eigen-solutions as the dimension of K and M.

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Eigen Analysis
Orthogonality where $\lambda_a \neq \lambda_b$

Following is the standard proof that the eigenvectors are orthogonal, or can be selected to be orthogonal to each other with respect to the mass and the stiffness matrix. Say:

- **that** λ_a **and** x_a **are an eigen pair to the** M **and** K **system**
- **that** λ_b **and** x_b **are another eigen pair to the** M **and** K **system**
- **and that** $\lambda_a \neq \lambda_b$

Then $\lambda_a M x_a - K x_a = 0$ **and** $\lambda_b M x_b - K x_b = 0$**.**

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Eigen Analysis
Orthogonality where $\lambda_a \neq \lambda_b$

$\lambda_a M x_a - K x_a = 0$ **and** $\lambda_b M x_b - K x_b = 0$**.**

Observe that:

1. $\lambda_a x_b^T M x_a - x_b^T K x_a = 0$

2. $\lambda_b x_a^T M x_b - x_a^T K x_b = 0$

Subtracting one from the other, and exploiting symmetry of M **and** K**, we see that** $(\lambda_a - \lambda_b) x_a^T M x_b = 0$**. Since** $\lambda_a \neq \lambda_b$**,** $x_a^T M x_b = 0$**.**

From either 1.) or 2.), we see also that $x_a^T K x_b = 0$**.**

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Eigen Analysis
Orthogonality where $\lambda_a = \lambda_b$

Say:

- that λ_a and x_a are an eigen pair to the M and K system
- that λ_b and x_b are another eigen pair to the M and K system
- and that $\lambda_a = \lambda_b = \lambda$ but that $x_a \neq x_b$

Then $\lambda_a M x_a - K x_a = 0$ and $\lambda_b M x_b - K x_b = 0$.

Define $\bar{x}_a = x_a - c x_b$ where $c = (x_a^T M x_b)/(x_b^T M x_b)$.

Note that $(\lambda, \bar{x}_a)$ is still an eigen-solution, that $\bar{x}_a^T M x_b = 0$, and that $\bar{x}_a^T K x_b = 0$.

We have a new, equivalent, but orthogonal pair of eigen vectors.

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Eigen Analysis
Orthogonality and Normalization

The equation $\lambda Mx - Kx = 0$ leaves the vector x undetermined in magnitude. If (λ, x) works, then so does $(\lambda, \alpha x)$ for any scalar α.

We normalize the eigenvectors so that $x_k^T M x_k = 1$ for each eigen vector x_k. This is *Mass Normalization*. These vectors are *orthonormal* with respect to the mass matrix.

Note that because $\lambda_k M x_k - K x_k = 0$, $x_k^T K x_k = \lambda_k$

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Eigen Analysis
Significance of Orthogonality

Lets consider the solution $x(t)$ to $M\ddot{x}(t) + Kx(t) = F(t)$ and expand $x(t) = \sum_{k=1}^{N} \beta_k(t) x_k$ where x_k is the k'th normalized eigenvector, corresponding to the k'th eigen value λ_k.

We note that this expression is actually a change of variables $\{x\} = P\{\beta\}$ where the columns of P are the normalized eigenvectors. See what this does to the mass and stiffness matrices.

$$P^T M P = I \quad \text{and} \quad P^T K P = \Lambda$$

where I is the N by N identity matrix and Λ is the diagonal matrix of eigenvalues. Because M and K are positive definite, $0 < \lambda_k$ for each k.

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Eigen Analysis

The matrix P is called the MODAL MATRIX.

If the eigenvalues are all unique, and the columns are ordered according to the order of the eigen values, then the modal matrix is unique.

If there are repeated eigenvalues, then the columns corresponding to repeated eigenvalues in one version of the modal matrix must span the space of the corresponding columns in any other version.

The β_k are called MODAL COORDINATES. We shall use them quite a lot as generalized coordinates.

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Eigen Analysis
System Solution

In modal coordinates, the governing equation becomes

$$\ddot{\beta} + \Lambda\beta = P^T F = \tilde{F}$$

We can solve for these modal coordinates, β_k, individually. We can solve for them numerically, and sometimes we can achieve analytic solutions.

The solution in physical coordinates can be recovered from $x(t) = P\,\beta(t)$

Lets examine free vibration: $\ddot{\beta} + \Lambda\beta = 0$**. This has solution**

$\beta_k(t) = Re\left\{ A_k e^{i\sqrt{\lambda_k}t} \right\}$**. We now identify the eigenvalues in terms of natural frequencies:** $\lambda_k = \omega_k^2$**.**

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Eigen Analysis
Another Motivation

We have motivated the search for eigen-solutions by looking for the congruence transform that would diagonalize both the mass and the stiffness matrix.

We can also motivate the search for eigen-pairs by asking ourselves:

Given the system $M\ddot{x} + Kx = 0$, with some set of initial conditions, can we express the solution as $x(t) = \sum_{k=1}^{N} \beta_k(t) x_k$ for some set of basis vectors x_k?

The answer is YES, and we have demonstrated the character of the eigenvectors x_k in the proceeding slides.

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Eigen Analysis
Another Motivation

Given $M\ddot{x} + Kx = 0$**, postulate a solution** $x(t) = \sum_{k=1}^{N} \beta_k(t) x_k$

From which $\sum_{k=1}^{N} \ddot{\beta}_k(t) M x_k + \beta_k(t) K x_k$**. Consider the case where**

$\beta_m(0) = 0$ **for all** $m \neq k$**, then** $\ddot{\beta}_k(t) M x_k + \beta_k(t) K x_k = 0$**.**

We expect a solution $\beta_k(t) = Re\left\{ A_k e^{i\omega_k t} \right\}$**, which yields**

$-\omega_k^2 M x_k + K x_k = 0$**: the algebraic eigenvalue problem.**

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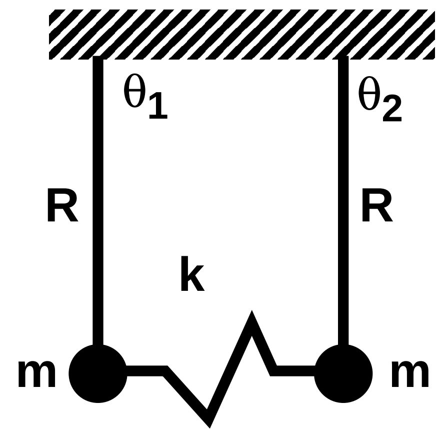
Example Problem

Consider the system of two identical, freely -hanging pendula shown. There are two degrees of freedom, so the mass an stiffness matrices will each be 2x2, and there will be exactly two eigen-pairs.



Lets first derive the linearized governing equations: $T = \frac{m}{2}R^2\dot{\theta}_1^2 + \frac{m}{2}R^2\dot{\theta}_2^2,$

$$V = -mgR\cos\theta_1 - mgR\cos\theta_2 + \frac{k}{2}R^2(\sin\theta_2 - \sin\theta_1)^2$$

to find $\begin{bmatrix} mR^2 & 0 \\ 0 & mR^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} mgR + kR^2 & -kR^2 \\ -kR^2 & mgR + kR^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = 0$

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Example Problem

Lets look for solutions $\left(\omega^2, \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}\right)$ **to the eigen-problem:**

$$-\omega^2 \begin{bmatrix} mR^2 & 0 \\ 0 & mR^2 \end{bmatrix} \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + \begin{bmatrix} mgR + kR^2 & -kR^2 \\ -kR^2 & mgR + kR^2 \end{bmatrix} \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = 0$$

In dimensionless form, this is

$$-\alpha^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + \begin{bmatrix} 1+\gamma & -\gamma \\ -\gamma & 1+\gamma \end{bmatrix} \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = 0$$

where $\alpha^2 = \omega^2 R/g$ **and** $\gamma = Rk/mg$

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Example Problem

Combine the two matrices

$$\begin{bmatrix} 1+\gamma-\alpha^2 & -\gamma \\ -\gamma & 1+\gamma-\alpha^2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = 0$$

A non-trivial solution requires the determinant to be zero:

$$(1+2\gamma) - 2\alpha^2(1+\gamma) + \alpha^4 = 0.$$

This yields solutions $\alpha^2 = 1+\gamma \pm \gamma = 1, 1+2\gamma$.

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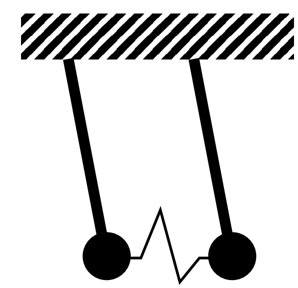
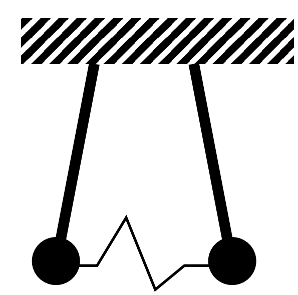
Example Problem

We substitute these back into the matrix equation to solve for θ_1 & θ_2:



$$\begin{bmatrix} \gamma & -\gamma \\ -\gamma & \gamma \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = 0$$ **for** $\alpha^2 = 1$**, from which we see** $\theta_1 = \theta_2$**. This is the pendula swinging in unison.**



$$\begin{bmatrix} -\gamma & -\gamma \\ -\gamma & -\gamma \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = 0$$ **for** $\alpha^2 = 1 + 2\gamma$**, from which we see** $\theta_1 = -\theta_2$**. This is the pendula swinging counter to each other.**

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Example Problem

Lets normalize these modes with respect to the mass matrix to calculate our transformation matrix:

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2$$ **=> we must scale this mode:** $$x_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2$$ **=> we must scale this mode:** $$x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

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Example Problem

If we guess the eigen-modes first, then we can deduce the eigenvalues from the eigen equation.

Given $x_1 \sim \begin{bmatrix} 1 \\ 1 \end{bmatrix}$**, and** $\begin{bmatrix} 1+\gamma-\alpha^2 & -\gamma \\ -\gamma & 1+\gamma-\alpha^2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = 0$**, we find**

$$\begin{bmatrix} 1-\alpha^2 \\ 1-\alpha^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha_1^{\ 2} = 1.$$

Similarly, given $x_2 \sim \begin{bmatrix} 1 \\ -1 \end{bmatrix}$**, we find** $\begin{bmatrix} 1+2\gamma-\alpha^2 \\ -1-2\gamma+\alpha^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **=>**

$$\alpha_2^{\ 2} = 1+2\gamma.$$

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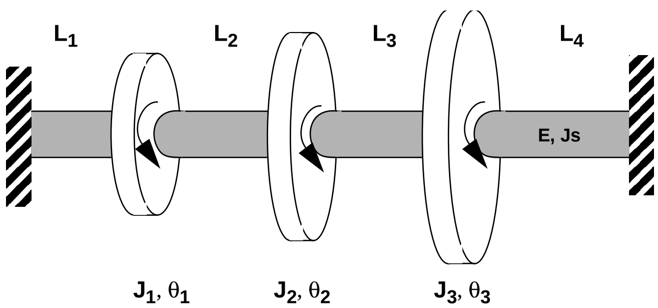
Homework
Due Monday



Lets consider a structure similar to the disk-shaft system in Meirovitch.

Ignoring the kinetic energy of the shaft:

- **Derive the linear equations for this system**
- **Write the matrix eigen-equation**
- **Assuming that** $J_1 = J_2 = J_3$ **and that** $L_1 = L_2 = L_3 = L_4$**, write a dimensionless version of the matrix eigen equation.**
- **Using Matlab, find the eigen solutions.**
- **Plot modes as in Fig 4.1.**
- **Write the modal matrix. Remember to normalize w.r.t. mass matrix.**

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What We Might Have Learned Up to Now

Governing Equations For Assembled Systems

Complex Representation For Force And Displacement In Steady State

RMS Quantities And Energy Dissipation In Steady State

Impulse Response Functions

Laplace Solution To Simple Systems

Lagrange Equations

Calculation Of Generalized Forces By Virtual Work And Potential Energy Of Loading

Assumed Modes Method

Conservation Of Mechanical Energy

Linearization

A Little About Eigen-analysis

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