Slides of Lecture 3

Today's Class

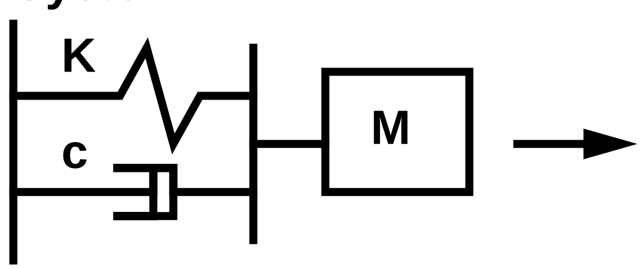
- **Homework**
- **Fourier Analysis and the Complex Operators**

Homework Number 2

Part A. Determine the impulse response function $h(t)$ for System A. Assume that the system is under-damped ($\zeta < 1$).

Part B. Determine the indicial response function $g(t)$ for System A.

System A



Solution to Homework 2
Part A

The impulse response function satisfies

$$u(t) = \int_0^t h(t-\tau)F(\tau)d\tau \; .$$

By Laplace Transforms, $h(t) = \frac{1}{M}\mathcal{L}^{-1}\left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$

Our task is to invert this laplace transform.

Solution to Homework 2
Part A

We begin by noting that $s^2 + 2\zeta\omega_n s + \omega_n^2 = (s - r_1)(s - r_2)$

where $r_1 = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2}$ **and** $r_2 = -\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2}$

Note further that these are complex conjugates. This is no surprise since the roots of a polynomial of real coefficients are combinations of real numbers and complex conjugate pairs.

We now expand as partial fractions

$$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{(s - r_1)(s - r_1^*)} = \frac{A}{(s - r_1)} + \frac{B}{(s - r_1^*)}$$

We solve for A **and** B**:** $1 = A(s - r_1^*) + B(s - r_1)$ **to find**

$A = 1/2iIm(r_1)$ **and** $B = -1/2iIm(r_1)$

Solution to Homework 2
Part A

$$
\begin{aligned}
h(t) &= \frac{1}{M}\mathcal{L}^{-1}\left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \\
&= \frac{1}{M2iIm(r_1)}\mathcal{L}^{-1}\left\{\frac{1}{s-r_1} - \frac{1}{s-r_1{}^*}\right\} \\
&= \frac{1}{M2iIm(r_1)}\left\{e^{r_1 t} - e^{r_1{}^* t}\right\} \\
&= \frac{e^{Re\{r_1\}t}}{M2iIm(r_1)}\left\{e^{iIm\{r_1\}t} - e^{iIm\{r_1{}^*\}t}\right\} \\
&= \frac{e^{-\omega_n\zeta t}}{M\omega_n\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t)
\end{aligned}
$$

Partial Solution to Homework 2
Part B

Here we begin by recalling that

$$u(t) = \int_0^t g(t-\tau)\dot{F}(\tau)\,d\tau + F(0)g(t)$$

By taking the Laplace transform:

$$\begin{aligned} \mathcal{L}\{u\} &= \mathcal{L}\{g\}(s\mathcal{L}\{F\} - F(0)) + \mathcal{L}\{g\}F(0) \\ &= s\mathcal{L}\{g\}\mathcal{L}\{F\} \end{aligned}$$

but we recall that $\mathcal{L}\{u\} = \mathcal{L}\{h\}\mathcal{L}\{F\}$,

and we conclude that $\mathcal{L}\{g\} = \mathcal{L}\{h\}/s$

Partial Solution to Homework 2
Part B

For this problem,

$$\mathcal{L}\{g\} = \frac{1}{s}\mathcal{L}\{h\} = \frac{1}{M2iIm(r_1)}\frac{1}{s}\left\{\frac{1}{s-r_1} - \frac{1}{s-r_1{}^*}\right\}$$

We expand this into partial fractions (yuck):

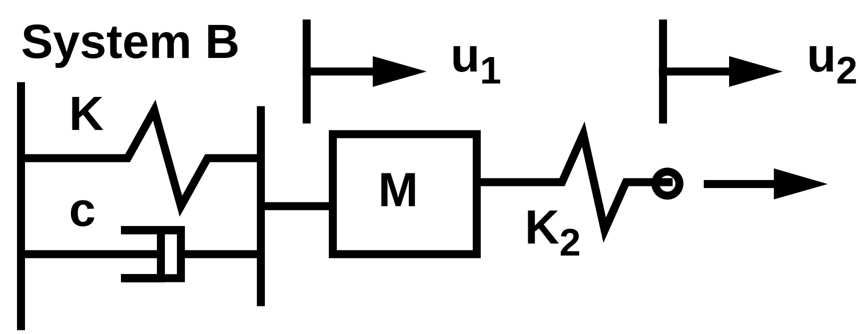
$$\frac{1}{s}\left\{\frac{1}{s-r_1} - \frac{1}{s-r_1{}^*}\right\} = \left[-\frac{1/r_1}{s} + \frac{1/r_1}{s-r_1} + \frac{1/r_1{}^*}{s} - \frac{1/r_1{}^*}{s-r_1{}^*}\right]$$

It is up to the students to complete the assignment.

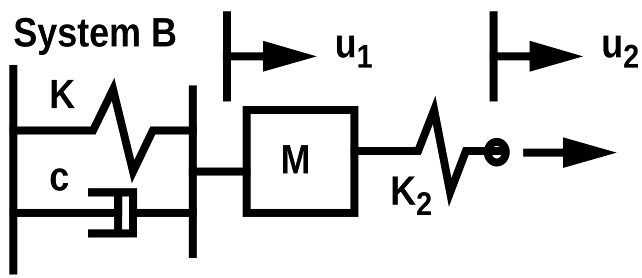
Solution to Homework 3
Part A

Part A. Resolve out the degree of freedom u1 associated with the mass in System B to in favor of u2. Obtain a governing equation in a single degree of freedom u2. You may leave this in Laplace transform space.

Part B. Determining the impulse response function for this problem. This function lives in the time domain. Assume that the system is under-damped.



Solution to Homework 3
Part A



$$M\ddot{u}_1 + c\dot{u}_1 + Ku_1 = K_2(u_2 - u_1)$$

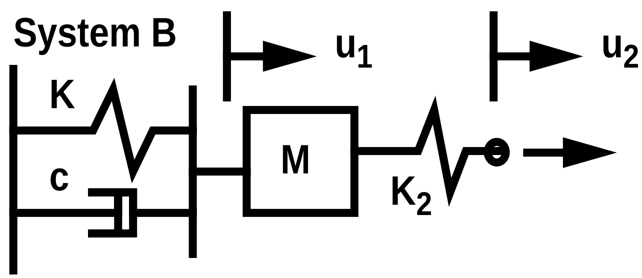
$$F = K_2(u_2 - u_1)$$

Lets rewrite the first equation as $M\ddot{u}_1 + c\dot{u}_1 + Ku_1 = F$ **which is solved by Laplace transform:**

$$\mathcal{L}\{u_1\} = \frac{\mathcal{L}\{F\}}{M(s^2 + 2\zeta\omega_n + \omega_n^2)}$$ **so** $u_1 = h_1 * F$

where $$h_1(t) = \frac{e^{-\omega_n \zeta t}}{M\omega_n\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t)$$

Solution to Homework 3
Part A



$$M\ddot{u}_1 + c\dot{u}_1 + Ku_1 = F$$

$$F = K_2(u_2 - u_1)$$

We solve the second equation for u_2:

$$u_2 = \frac{F}{K_2} + u_1$$

$$= \frac{F}{K_2} + h_1 * F \qquad \Rightarrow h_2 = h_1 + \frac{1}{K_2}\delta$$

$$= \left(h_1 + \frac{1}{K_2}\delta\right) * F$$

Concepts from Complex Variable Analysis
A Reminder

General Notions

Complex number $z = x + iy$

Real and Imaginary operators: $Re\{x + iy\} = x$, $Im\{x + iy\} = y$

Complex Conjugate: $(x + iy)^* = x - iy$

Observe: $Re\{z\} = \frac{1}{2}(z + z^*)$ **and** $Im\{z\} = \frac{1}{2i}(z - z^*)$

Complex Function of a Complex Variable: $f(z) = u(z) + iv(z)$

An Analytic Function has a Unique Derivative:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial (iy)} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Concepts from Complex Variable Analysis
A Reminder

Examples of Analytic Functions

$z = x + iy$ **is analytic**

z^n **is analytic if n is an integer**

$e^{\alpha z}$**,** $\sin(\alpha z)$ **and** $\cos(\alpha z)$ **are analytic**

Identifying Analytic Functions, Some Useful Rules

if f_1 and f_2 are each analytic, then so are

$$\alpha f_1,\ f_1 + f_2,\ \ f_1 f_2\ ,\ f_1 / f_2$$

if f is analytic, then so is f'

if $Re(f)$ is constant then $Im(f)$ is also constant.

if $Im(f)$ is constant then $Re(f)$ is also constant.

Analytic Functions, Some More Useful Rules

The $Re\{\ \ \}$ operator distributes over addition: (trivially.)

$$Re\{A+B\} = Re\{A\} + Re\{B\}$$

If $f(\alpha)$ is a complex function of a real parameter, α, then the Re and derivative operator commute: (a little harder to see.)

$$\frac{d}{d\alpha}Re\{f(\alpha)\} = Re\left\{\frac{df}{d\alpha}\right\}$$

Analytic Functions, Some More Useful Rules

The integral over a cycle of two harmonic functions: (more difficult)

$$\frac{\omega}{2\pi}\int_0^{\frac{\omega}{2\pi}} Re\{Ae^{i\omega t}\}Re\{Be^{i\omega t}\}dt = \frac{1}{2}Re\{AB^*\}$$

$$= \frac{1}{2}[Re\{A\}Re\{B\} + Im\{A\}Im\{B\}]$$

Analytic Functions, Some More Useful Rules

When you have the trick to prove the above, you can show the following:

$$\frac{\omega_0}{2\pi}\int_0^{\frac{\omega}{2\pi}} Re\left\{\sum_{p=0}^{\infty} A_p e^{i\omega_0 pt}\right\} Re\left\{\sum_{q=0}^{\infty} B_q e^{i\omega_0 qt}\right\} dt$$

$$= Re\{A_0 B_0^*\} + \frac{1}{2}\sum_{q=1}^{\infty} Re\{A_q B_q^*\}$$

About Real (Actual) Forces and Displacements

Forces and displacements are real. They may be scalar, vector, or algebraic, but they are not complex. On the other hand, we sometimes define complex quantities to represent, indirectly, periodic forces:

$$f(t) = Fe^{i\omega t}$$ **in NOT a force**

The following could be forces:

$$Re\{Fe^{i\omega t}\}, Im\{Fe^{i\omega t}\}, Re\{Fe^{-i\omega t}\}, Im\{Fe^{-i\omega t}\}$$

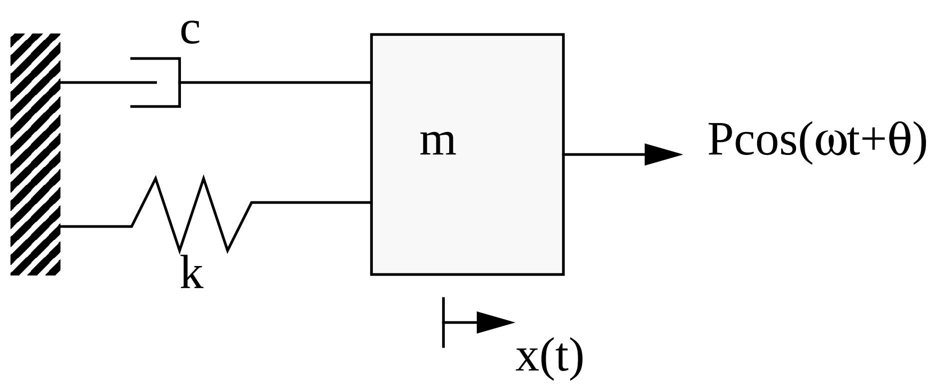
or $$Fe^{i\omega t} + F^* e^{-i\omega t}$$

This last form will be especially useful when we deal with complex modes.

In the above, F could be a complex scalar, a complex vector, an array of complex scalars, or an array of complex vectors. We will consider the more complicated cases later in this course.

Application to Damped Harmonic Oscillator

Consider the simple, single-degree-of-freedom oscillator shown below:



The equation of motion for the mass is:

$m\ddot{x} + c\dot{x} + kx = P\cos(\omega t + \theta)$ **which we can express in exponential form by**

$$P\cos(\omega t + \theta) = Re\{Fe^{i\omega t}\}$$

where $Re\{F_1\} = P\cos(\theta)$ **and** $Im\{F_1\} = P\sin(\theta)$**.**

Solution in Frequency Space

Lets postulate that the steady state displacement has the same form:

$$u(t) = Re\{Xe^{i\omega t}\}$$

In order to evaluate the time derivatives of x, we must observe that the $Re\{\ \}$ and derivative operators commute:

$$\dot{x}(t) = \frac{d}{dt}Re\{Xe^{i\omega t}\} = Re\left\{\frac{d}{dt}Xe^{i\omega t}\right\} = Re\{i\omega Xe^{i\omega t}\}$$

and similarly, $\ddot{x}(t) = Re\{-\omega^2 Xe^{i\omega t}\}$

Substituting back into the equation of motion:

$$mRe\{-\omega^2 Xe^{i\omega t}\} + cRe\{i\omega Xe^{i\omega t}\}$$
$$+ kRe\{Xe^{i\omega t}\} - Re\{Fe^{i\omega t}\} = 0$$

We observe that the Re operator distributes over addition:

$$Re\{[(-\omega^2 m + i\omega c + k)X - F]e^{i\omega t}\} = 0$$

There are several easy ways to prove that the complex argument of the Re operator in the above equation is zero. Then the coefficient of $e^{i\omega t}$ is also zero:

$$[(-\omega^2 m + i\omega c + k)X - F] = 0$$

Solution

Solving for $X(\omega)$ from which we find that

$$X = \frac{F}{(-\omega^2 m + i\omega c + k)}$$

$$= \frac{F/k}{1 - (\omega/\omega_n)^2 + 2i\zeta(\omega/\omega_n)} = \frac{F}{k} H(\omega)$$

where $\omega_n^2 = k/m$ and $\zeta = c/(2m)$. $H(\omega)$ is the magnification factor.

The text discusses the magnification factor, its magnitude, where the magnitude is largest, and the role of damping - ζ - in calculating Q

Limiting Cases

Let's examine limiting cases:

$\omega/\omega_n \to 0 \Rightarrow F = kx$**: this is statics, force is in phase with displacement**

$\omega/\omega_n \to \infty \Rightarrow F = -xm\omega^2$**: mass loading is opposite to displacement**

$$\omega = \omega_n \Rightarrow x = \frac{1}{2}\left[1 - i\frac{\sqrt{1-\zeta^2}}{\zeta}\right]F:$$

For small damping, $F = 2\zeta iX$

Products and Integrals

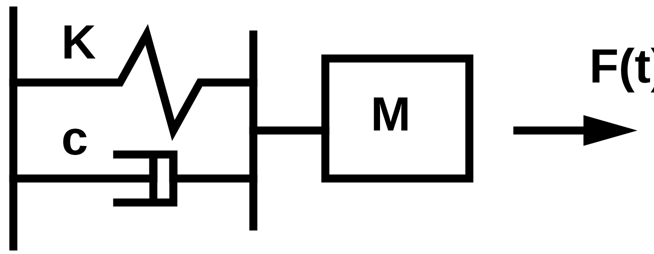
The importance of the rigor introduced above is illustrated in the calculation of the time-averaged rate of energy dissipation.

$$
\begin{aligned}
D &= \frac{1}{T}\int_0^T \dot{x}(\tau) f(\tau) d\tau \\
&= \frac{1}{T}\int_0^T Re\left\{ i\omega\frac{F}{k}He^{i\omega\tau}\right\} Re\{Fe^{i\omega\tau}\} d\tau \\
&= -\frac{\omega}{2k} Im(H)|F|^2
\end{aligned}
$$

where $T = 2\pi/\omega$**.**

Fourier Analysis

System A



We again consider the damped linear oscillator. This time we consider an applied force

$$F(t) = Re\left\{ \sum_{p=0}^{\infty} C_p e^{ip\omega_0 t} \right\}$$

(The "Re" operator was neglected in equations 1.86, 1.87of the text.)

We deduce the coefficients of the Fourier series from integration against the basis functions:

$$C_0 = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} F(t)dt \text{, and } C_q = \frac{\omega_0}{\pi} \int_0^{2\pi/\omega_0} F(t)e^{-i\omega_0 q} dt \text{ for } 0 < q$$

Please verify this yourselves!

Response of the System

Exploiting linearity and our previous analysis of damped linear systems,

$$x(t) = Re\left\{ \sum_{p=0}^{\infty} (C_p/k) H(\omega_p) e^{ip\omega_0 t} \right\}$$

where $H(\omega) = \dfrac{1}{[1-(\omega/\omega_n)^2] + i2\zeta(\omega/\omega_n)}$

This is the key result of Fourier Analysis.

Let's Calculate some Averages The Easy Way: Re-Examine Page 16

Lets calculate the square of RMS force:

$$\bar{F}^2 =$$

$$\frac{\omega_0}{2\pi}\int_0^{\frac{\omega}{2\pi}} Re\left\{\sum_{p=0}^{\infty} C_p e^{i\omega_0 pt}\right\} Re\left\{\sum_{q=0}^{\infty} C_q e^{i\omega_0 qt}\right\} dt$$

$$= C_0^{\ 2} + \frac{1}{2}\sum_{q=0}^{\infty} C_q C_q{}^*$$

$$= C_0^{\ 2} + \frac{1}{2}\sum_{q=0}^{\infty} |C_q|^2$$

Let's Calculate some Averages The Easy Way: Re-Examine Page 16

Lets calculate the square of RMS displacement:

$$\bar{x}^2 =$$

$$\frac{\omega_0}{2\pi}\int_0^{\frac{\omega}{2\pi}} Re\left\{\sum_{p=0}^{\infty}\frac{C_p}{k}H(\omega_p)e^{i\omega_0 pt}\right\} Re\left\{\sum_{q=0}^{\infty}\frac{C_q}{k}H(\omega_q)e^{i\omega_0 qt}\right\} dt$$

$$= \left(\frac{C_0}{k}\right)^2 + \frac{1}{2k^2}\sum_{q=0}^{\infty} C_q C_q{}^* H(\omega_q) H(\omega_q)^*$$

$$= \left(\frac{C_0}{k}\right)^2 + \frac{1}{2k^2}\sum_{q=0}^{\infty} |C_q|^2 |H(\omega_q)|^2$$

Let’s Calculate some Averages The Easy Way: Re-Examine Page 16

Lets calculate the square of RMS dissipation rate:

$$\overline{F\dot{x}} =$$

$$\frac{\omega_0}{2\pi}\int_0^{\frac{\omega}{2\pi}} Re\left\{\sum_{p=0}^{\infty} C_p e^{i\omega_0 pt}\right\} Re\left\{\sum_{q=1}^{\infty} i\omega_0 q \frac{C_q}{k} H(\omega_q) e^{i\omega_0 qt}\right\} dt$$

$$= -\frac{1}{2k}\sum_{q=0}^{\infty} C_q C_q{}^{*} Im\{H(\omega_q)\}$$