Lecture 18: Numerical Methods

Numerical Solutions for Rods
Galerkin Method

We saw how we could use the eigen functions to convert the partial differential equations into systems of ordinary differential equations that could be solved by expressing the displacement field a a linear combination of eigen modes.

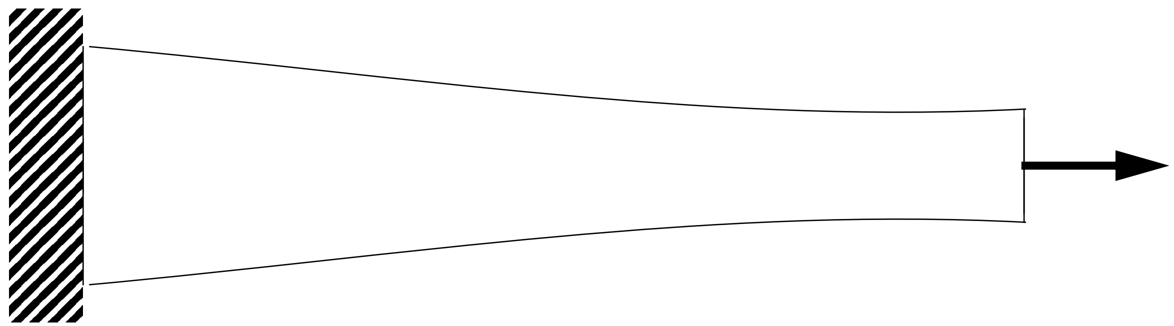
Application of the Lagrange equations resulted in a system of ordinary differential equations in time - one equation per modal coefficient.

In this lecture, we discuss the solution of vibration problems when we do not have a closed-form expression for the eigen modes. There are two general approaches

- **numerical solution to approximate the eigen functions**
- **solution for displacements with going through eigen analysis**

Numerical Solution For Eigen Functions

Say that we have a problem for which we cannot calculate closed form eigen functions. One such problem is that pictured below where the



cross section is non-constant. We shall now expand the displacement field in terms of assumed modes

$$u(x, t) = \sum_n a_n(t) P_n(x),$$

and then employ this expansion in creating the Lagrange equations. The only requirements that we place on $P_n(x)$ are that each satisfy what ever geometric boundary conditions exist.

Governing Equations

$$T(t) = \frac{1}{2}\int_0^L \rho A(x)\left(\sum_m \dot{a}_m(t)P_m(x)\right)\left(\sum_n \dot{a}_n(t)P_n(x)\right)dx$$

$$V(t) = \frac{1}{2}\int_0^L EA(x)\left(\sum_m a_m(t)P_m{}'(x)\right)\left(\sum_n a_n(t)P_n{}'(x)\right)dx$$

and

$$\delta W = \sum_n \delta a_n(t)P_n(L)F(t) = \sum_n \delta a_n(t)Q_n$$

Governing Equations

We have done this derivation before. The following should be familiar.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{a}_m}\right) + \frac{\partial V}{\partial a_m} = \sum_n M_{mn}\ddot{a}_n + \sum_n K_{mn}a_n = Q_m(t)$$

where $K_{mn} = \int_0^L EA(x)P_m{'}(x)P_n{'}(x)dx$**,**

$M_{mn} = \int_0^L \rho A(x)P_m(x)P_n(x)dx$**, and**

$$Q_m(t) = \int_0^L P_m(x)F(x,t) = P_m(L)F(t)$$

Governing Equations
How to choose the P_m

Criteria for choosing the assumed modes:

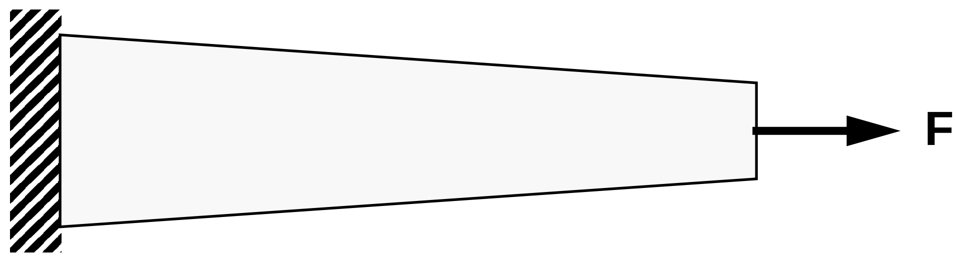
- **linear combinations of the assumed modes must be able to take on the admissible configurations of the problem. (They must span the space.)**
- **it should not be too difficult to compute the generalized coordinates and their time derivatives in terms of the initial deformation and velocity of the structure.**

Lets try some examples

Example with polynomial shape functions

Consider a tapered rod having cross section distribution

$$A(x) = A_0\left(1 - \frac{x}{2L}\right)$$



Assumed Modes Case 1

Lets try the expansion defined by $P_n(x) = (x/L)^n$

$$M_{nm} = \int_0^L \rho A_0 \left(1 - \frac{x}{2L}\right)(x/L)^m (x/L)^n dx$$

$$= \frac{\rho A_0 L}{2} \frac{(m+n+3)}{(m+n+1)(m+n+2)}$$

$$K_{mn} = \frac{mn}{L^2} \int_0^L E A_0 \left(1 - \frac{x}{2L}\right)(x/L)^{m+n-2} dx$$

$$= \frac{E A_0}{2L} \left(\frac{(mn)(m+n+1)}{(m+n-1)(m+n)}\right)$$

Assumed Modes Case 1

We have established that in this case, we can do the necessary integrals.

The question remains: how to find initial values: $a_n(0)$ **and** $\dot{a}_n(0)$**?.**

The answer is through the use of Bernstein polynomials.

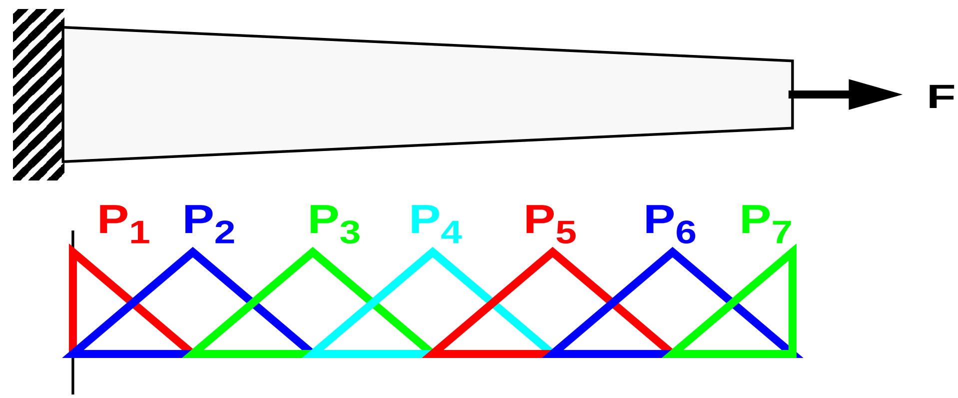
$$B_n(y(x,0)) = \sum_0^n y\left(L\frac{k}{n}, 0\right)\binom{n}{k}\left(\frac{x}{L}\right)^k\left(1-\left(\frac{x}{k}\right)\right)^{n-k} \rightarrow y(x,0)$$

$$B_n(\dot{y}(x,0)) = \sum_0^n \dot{y}\left(L\frac{k}{n}, 0\right)\binom{n}{k}\left(\frac{x}{L}\right)^k\left(1-\left(\frac{x}{k}\right)\right)^{n-k} \rightarrow \dot{y}(x,0)$$

We have established here that polynomials could be used to solve rod problems. This approach is correct, but not very appealing.

Assumed Modes Case 2

Lets use the the chapeau functions as our assumed modes



Each of these functions has

- a value of 1.0 on a single node,
- a non-zero value on a small region about its node (the support of the shape function),
- and a value of zero on every other node.

The approximation for $y(x, t)$ is $y(x, t) = \sum_n y_n(t) P_n(x)$

Assumed Modes Case 2

Initial conditions are obtained by several alternate methods, the most direct of which is collocation:

$y_n(0) = y(x_n, 0)$ **and** $\dot{y}_n(0) = \dot{y}(x_n, 0)$

The integrals $M_{nm} = \int_0^L \rho A_0 \left(1 - \frac{x}{2L}\right) y_n(x) y_m(x) dx$ **and**

$K_{nm} = \int_0^L E A_0 \left(1 - \frac{x}{2L}\right) y_n(x) y_m(x) dx$ **are calculated by numerical integration.**

The integration need be performed only where the support of $y_m(x)$ **intersects that of** $y_n(x)$