Lecture 15

TODAY

Last Word on Modal Analysis
Frequency Domain Analysis

A Last Word on Modal Analysis

Why modal analysis is attractive

- **modal coordinates can be integrated independently.**
- **Model reduction by modal truncation:**

$$x(t) = \sum_{k=1}^{M<modes} \beta_k(t)\{P_k\}$$

where $\{P_k\}$ is the k'th column of the modal matrix simplifies the problem further and permits larger time steps in the integration.

- **Vibration modes have intrinsic meaning. Modes and modal frequencies tell us something more understandable about structural response than do structural matrices**
- **Modes and modal frequencies can be measured and compared with prediction. Model validation and system identification**

Frequency Domain Analysis

We have already examined problems in the frequency domain when we assumed that the applied loads were harmonic:

$$f(t) = Re\{F_0 e^{i\omega t}\}.$$

In what follows, we will admit cases where the load is a continuous sum of harmonic inputs:

$$f(t) = \int_0^\infty Re\{F(\omega)e^{i\omega t}\}d\omega$$

Frequency Domain Analysis

We may also write this as

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

where $F(\omega)$ **has been defined a little differently than previously.**

There is a constraint that $F(-\omega) = (F(\omega))^*$ **.**

Lets discuss why.

Frequency Domain Analysis

If we can express the time domain function in terms of a frequency domain function, can we reverse the process?

Yes, of course. Following is a formulation of the relations.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega \quad \& \quad F(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

Note that we rescaled F in the above by 2π and replaced ω with $-\omega$.

There are several other conventions for the placement of the 2π term and the minus sign. The above appears to be most common.

These relations can be derived from a limiting case of Fourier series or from a single application of the Cauchy integral theorem.

Frequency Domain Analysis
An interesting but unimportant note

There is an interesting observation for fans of the Laplace transform. Fourier transform pairs can be achieved from analytic Laplace transforms where they exist:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt = \int_{0}^{\infty}[f(t)e^{i\omega t} + f(-t)e^{-i\omega t}]dt$$

$$= \mathcal{L}(f)\big|_{s = i\omega} + \mathcal{L}(-f)\big|_{s = -i\omega}$$

This is interesting, but not used very often. The primary reason is that one usually does not deal with analytic Fourier transforms.

Frequency Domain Analysis: Why?

Frequency domain analysis is most useful when comparison is to be made with experiment or when connection is desired between experiment and modal analysis. We shall see examples of each.

Frequency Domain Analysis: Governing Equation in Frequency Space

Our familiar governing equations are $M\ddot{x} + C\dot{x} + Kx = f(t)$.

The Fourier transform of this equation is

$$(-\omega^2 M + i\omega C + K)X(\omega) = F(\omega)$$

Say we are interested in the displacement $x_I(t)$ of degree of freedom I due to forces at degree of freedom J. At first look this would appear to be a difficult relationship to find, but it turns out to be reasonably do-able.

We diagonalize the governing equation via modal decomposition: $x(t) = P\beta(t)$ which is $X(\omega) = PB(\omega)$ in frequency space.

$$P^T(-\omega^2 M + i\omega C + K)PX(\omega) = P^T F(\omega)$$

Frequency Domain Analysis: Governing Equation in Frequency Space

$P^T(-\omega^2 M + i\omega C + K)PB(\omega) = P^T F(\omega)$ **becomes**

$$(-\omega^2 I + i2\omega\chi + D)B(\omega) = P^T F(\omega)$$

The terms of B can be solved individually:

$$B_k = \sum_{J=1}^{\mathrm{DOFs}} P_{Jk} F_J(\omega) / (\omega_k^2 - \omega^2 + 2i\omega\zeta_k)$$

Frequency Domain Analysis: Frequency Response Functions

The displacement at node I is

$$X_I(\omega) = \sum_{k=1}^{\text{Modes}} P_{Ik} B_k(\omega)$$

$$= \sum_{k=1}^{\text{Modes}} \sum_{J=1}^{\text{DOFs}} P_{Ik} P_{Jk} F_J(\omega) / (\omega_k^2 - \omega^2 + 2 i \omega \zeta_k)$$

$$= \sum_{J=1}^{\text{DOFs}} H_{IJ}(\omega) F_J(\omega)$$

Frequency Domain Analysis: Frequency Response Functions

Above,

$$H_{IJ}(\omega) = \sum_{k=1}^{\text{Modes}} P_{Ik}P_{Jk}/(\omega_k^2 - \omega^2 + 2i\omega\zeta_k)$$

is the “receptance frequency response function” of DOF I with respect to DOF J. Sometimes it is simply called the “frequency response function”.

By construction, we see that all of the frequency response functions for a single structure share the same peaks when plotted against frequency.

Lets discuss what this means for experimental modal analysis.

Frequency Domain Analysis: Modal Analysis

Having found the resonances of the system by identification of the peaks in the frequency response functions, we consider the following experiment:

We excite the structure at a single point J at a natural frequency ω_k and measure the displacements (or accelerations) at a good number of sample of points elsewhere on the structure

$$H_{IJ}(\omega) \cong P_{Ik}[P_{Jk}/(\omega_k^2 - \omega^2 + 2i\omega\zeta_k)]$$

We see that at resonance, the receptance at locations is proportional to the mode shape of that frequency. It is possible to assess the mode shapes experimentally.

Frequency Domain Analysis: Experimental Determination of $H_{IJ}(\omega)$

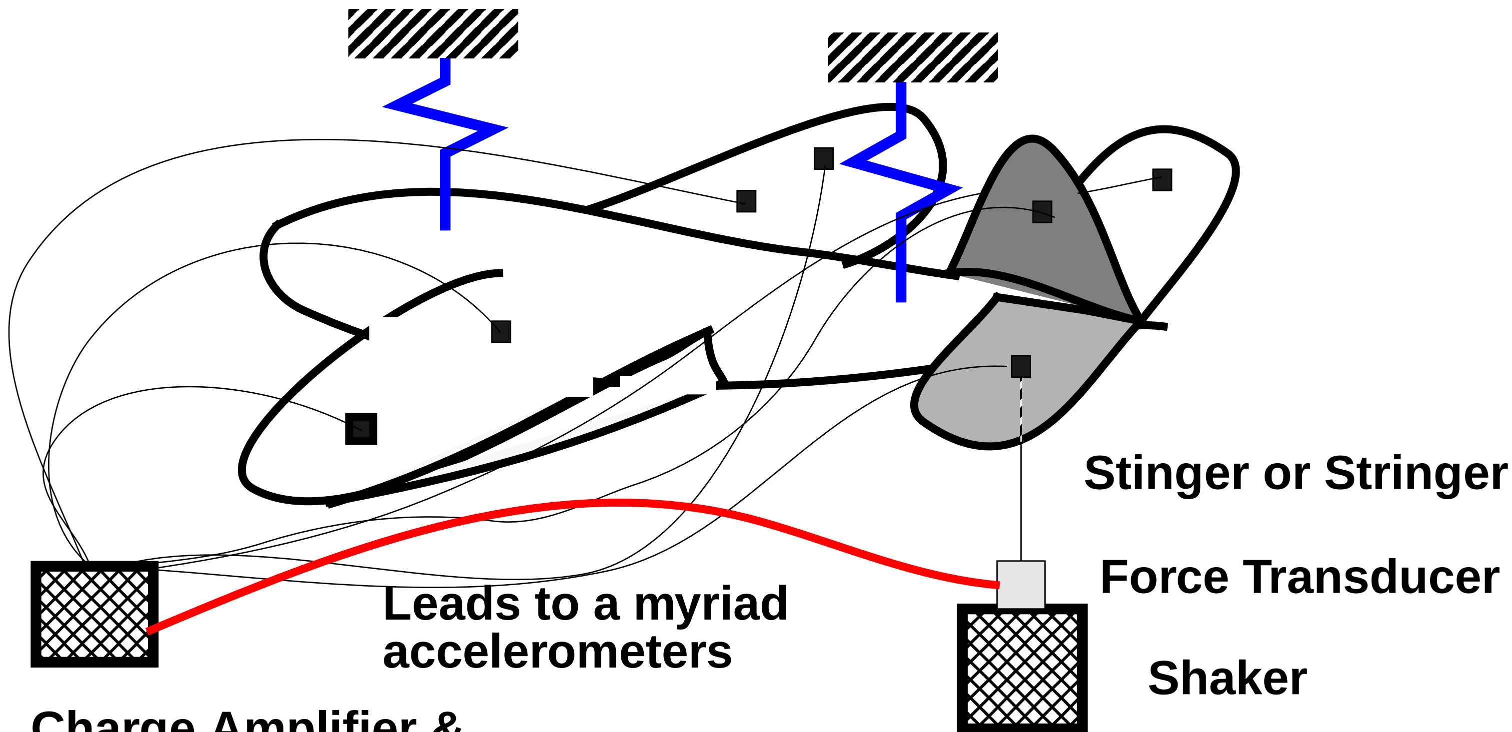
By definition, $H_{IJ}(\omega) = X_I(\omega)/F_J(\omega)$, and we will often want to determine $H_{IJ}(\omega)$ experimentally. What candidate forces can be used? Most commonly used are:

- **stepped sine**
- **swept sine**
- **impulse**
- **random**

Each approach has its advantages. Lets look at them individually

Experimental Determination of $H_{IJ}(\omega)$ By Stepped Sine

Stepped sine experiments are usually done by connecting a part of the structure to a shaker via a thin wire or rod (a stringer). Accelerometers placed at the appropriate locations on the structure provide the kinematic information in frequency space.



Experimental Determination of $H_{IJ}(\omega)$ By Stepped Sine

Stepped sine testing is usually done so that the structure reaches steady state at each frequency before moving on to the next frequency.

A sine force input of unit peak amplitude is

$$f(t) = \sin\omega_0 t = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}.$$

Since $$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega \quad ,$$

$$F(\omega) = \pi i[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Experimental Determination of $H_{IJ}(\omega)$ By Stepped Sine

Recalling that $H_{IJ}(\omega) = X_I(\omega)/F_J(\omega)$ and noting that the only information available from this experiment is at ω_0 and $-\omega_0$, we have

$H_{IJ}(\omega_0) = X(\omega_0)/(\pi i)$ **and** $H_{IJ}(-\omega_0) = X(\omega_0)^*/(-\pi i)$.

Note that $H_{IJ}(-\omega_0) = H_{IJ}{}^*(\omega_0)$. This means that $H_{IJ}(\omega)$ is the Fourier transform of a Real function.

Can we guess which one?

The advantage of stepped sine experiments is that they provide high resolution of both phase and amplitude of $H_{IJ}(\omega)$

The disadvantage is that it takes much more time.

Experimental Determination of $H_{IJ}(\omega)$ By Swept Sine Input

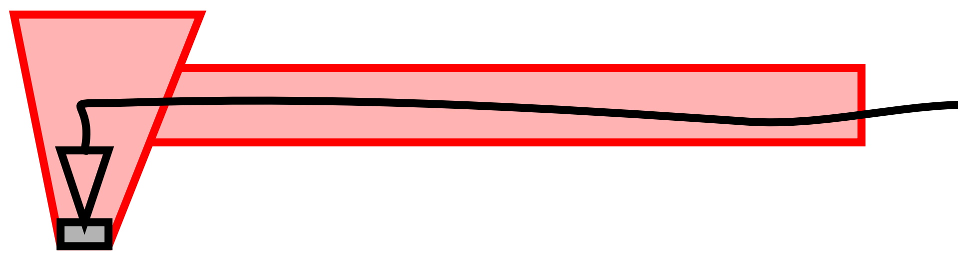
Swept sine input is much more quickly implemented than is the stepped sine input.

This sort of experiment is most often used to search for resonances.

Unless the sweep is very slow, the results are hard to post process for detailed information.

Experimental Determination of $H_{IJ}(\omega)$ By Impulse Hammer

The impulse hammer contains a precision force transducer at its tip so that the force history at the interval of impact can be recorded accurately. The frequency content of the impulse extends roughly to the



reciprocal of the period of the pulse. The frequency content can be concentrated at low frequencies by softening the tip and by increasing the mass of the hammer head. Conversely, the frequency content can be extended up to higher frequencies by dropping the mass and raising the stiffness.

The frequency content of the structural response will cover roughly the same interval.

Experimental Determination of $H_{IJ}(\omega)$ By Impulse Hammer

In principle, the frequency response can be assessed by evaluating the Fourier transforms of the impulse and the response and taking the ratio.

$$H_{IJ}(\omega) = X_I(\omega)/F_J(\omega)$$

Signals are reinforced by doing several experiments and averaging the frequency response functions obtained from each.

Note that the magnitude information on the frequency response function can be calculated from power spectra of the input and output:

$[H_{IJ}(\omega)F_J(\omega)][H_{IJ}(\omega)F_J(\omega)]^* = X_I(\omega)X_I(\omega)^*$ **yielding**

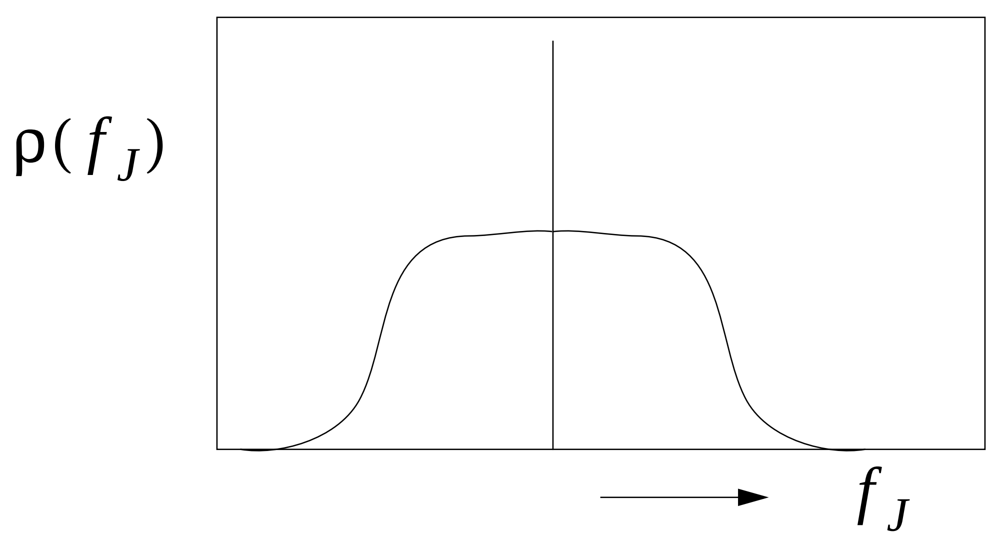
$$|H_{IJ}(\omega)| = \sqrt{S_{XX}(\omega)}/\sqrt{S_{FF}(\omega)}$$

where $S_{XX}(\omega) = X_I(\omega)X_I(\omega)^*$ **is the power spectral density (PSD) of displacement and** $S_{FF}(\omega) = F_J(\omega)F_J(\omega)^*$ **is the PSD of force.**

Experimental Determination of $H_{IJ}(\omega)$ Via Random Input

Random input experiments offer the opportunity to collect data rich in frequency content. Further, those experiments can be run over a long period of time to over acquiring a good signal-to-noise ratio.

We consider stationary random excitations. These are processes whose probability distributions are constant in time. Following is a plot of a probability density distribution of values of f_J determined from a population collected over a large time window. Stationary asserts that the shape of this curve does not change as the sampling window is translated in time.



Experimental Determination of $H_{IJ}(\omega)$ Via Random Input

Define the auto-correlation integral for the imposed force:

$u_{ff}(\tau) = \int_{-\infty}^{\infty} f_J(t) f_J(t+\tau) dt$. **Because the distribution of f_J is stationary, $u_{ff}(\tau)$ is well defined (does not change with a change of variable $s = t + \alpha$). Similarly, because the system is linear and time-translation invariant, the auto-correlation integral $u_{xx}(\tau)$ for the system response at location I is well defined as well.**

The cross-correlation integral is $u_{xf}(\tau) = \int_{-\infty}^{\infty} x_I(t) f_J(t+\tau) dt$.

Note that these infinite integrals should average away much of the noise in the system. Further, if the excitation is also ergodic, we can average the correlation integrals for multiple experiments.

Experimental Determination of $H_{IJ}(\omega)$ Via Random Input

In summary, the auto-correlation and cross-correlations can be collected and averaged in various ways to maximize the signal-to-noise.

The Fourier transforms of these integrals have interesting form:

$$U_{xf}(\omega) = \int_{-\infty}^{\infty} e^{i\omega\tau} u_{xf}(\tau)d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega\tau} x_I(t) f_J(t+\tau)dtd\tau$$

$$= \int_{-\infty}^{\infty} x_I(t)\left[\int_{-\infty}^{\infty} e^{i\omega\tau} f_J(t+\tau)d\tau\right]dt$$

$$= \left[\int_{-\infty}^{\infty} e^{-i\omega\tau} x_I(\tau)d\tau\right] F_J(\omega) = X_I(\omega)^* F_J(\omega)$$

Experimental Determination of $H_{IJ}(\omega)$ Via Random Input

Because our integrals are actually over finite time, we use the expected value of the right hand side.

$$S_{XF}(\omega) = E\{U_{xf}(\omega)\} = E\{X_I(\omega)^* F_J(\omega)\}.$$

where the expected value is obtained by averaging over a number of tests. The uncertainty in the expected value is a function of the number of averages. $S_{XF}(\omega)$ is the cross-spectral density of X_I and F_J.

Note that $S_{XF} = S_{FX}{}^*$.

Similarly,

$$S_{XX}(\omega) = E\{X_I(\omega)^* X_I(\omega)\} \text{ \& } S_{FF}(\omega) = E\{F_J(\omega)^* F_J(\omega)\}$$

are power spectral densities of X_I and F_J.

Experimental Determination of $H_{IJ}(\omega)$ Via the Time Domain

Contracting the equation

$$X_I(\omega) = H_{IJ}(\omega)F_J(\omega)$$

on both sides by the appropriate quantity Fourier transformed quantity, we see that we could as easily evaluate

$H_{IJ}(\omega) = S_{FX}(\omega)/S_{FF}(\omega)$ **and** $H_{IJ}(\omega) = S_{XX}(\omega)/S_{XF}(\omega)$

These equations offer two distinct estimates of $H_{IJ}(\omega)$ from experimental data. The ratio of these two estimates is the coherence

$$\gamma^2(\omega) = |S_{FX}(\omega)|^2/(S_{FF}(\omega)S_{XX}(\omega))$$

Unless γ^2 is nearly 1.0, the estimate for $H_{IJ}(\omega)$ is considered unreliable.

Next Time

Clay Fulcher will demonstrate how modal and frequency response are uses to assess reliability of critical structures.