Slides of Lecture 9

Another Example of Linearization

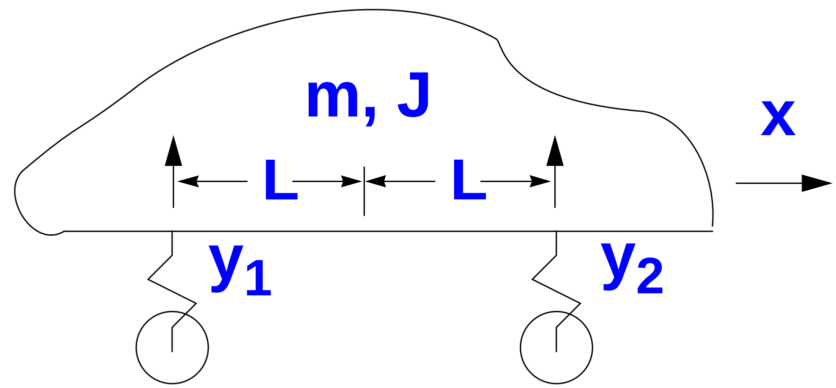
Another Example of Lagrange Equations with Generalized Degrees of Freedom

In-Class Practice of Problems of Students's Choice

More Linearization

Derive the governing equations for x, y_1, & y_2 for the car shown.

Then linearize this set of three equations.



More Linearization Solution

Define the angle θ so that $\sin\theta = \dfrac{y_2 - y_1}{2L}$. This is the rotation of the car about its c.g. Also, define the vertical displacement $y = (y_1 + y_2)/2$ of the c.g.

Express the kinetic energy: $T = \dfrac{m}{2}\dot{x}^2 + \dfrac{m}{2}\dot{y}^2 + \dfrac{J}{2}\dot{\theta}^2$

The potential energy is $V = mgy + \dfrac{k}{2}y_1^2 + \dfrac{k}{2}y_2^2$

Derive the governing equation for each of x, y_1, and y_2:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) + \frac{\partial V}{\partial x} = m\ddot{x} = 0$$

More Linearization Solution

$$\frac{\partial T}{\partial \dot{y}_1} = \left(\frac{\partial T}{\partial \dot{y}}\right)\left(\frac{\partial y}{\partial y_1}\right) + \left(\frac{\partial T}{\partial \dot{\theta}}\right)\left(\frac{\partial \theta}{\partial y_1}\right) + \left(\frac{\partial T}{\partial \dot{x}}\right)\left(\frac{\partial x}{\partial y_1}\right)$$

$$= \frac{m\dot{y}}{2} - \frac{J\dot{\theta}}{2L}\frac{1}{\sqrt{1-\left(\frac{y_2-y_1}{2L}\right)^2}}$$

$$= \frac{m(\dot{y}_1+\dot{y}_2)}{2} - \frac{J}{2L}\frac{(\dot{y}_2-\dot{y}_1)}{1-\left(\frac{y_2-y_1}{2L}\right)^2}$$

More Linearization Solution

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_1}\right) = \frac{m(\ddot{y}_1 + \ddot{y}_2)}{2} - \frac{J}{2L}\frac{(\ddot{y}_2 - \ddot{y}_1)}{1 - \left(\frac{y_2 - y_1}{2L}\right)^2}$$

$$-\frac{2J}{(2L)^2}\frac{(\dot{y}_2 - \dot{y}_1)^2}{\left[1 - \left(\frac{y_2 - y_1}{2L}\right)^2\right]^2}$$

and similarly

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_2}\right) = \frac{m(\ddot{y}_1 + \ddot{y}_2)}{2} + \frac{J}{2L}\frac{(\ddot{y}_2 - \ddot{y}_1)}{1 - \left(\frac{y_2 - y_1}{2L}\right)^2}$$

$$+\frac{2J}{(2L)^2}\frac{(\dot{y}_2 - \dot{y}_1)^2}{\left[1 - \left(\frac{y_2 - y_1}{2L}\right)^2\right]^2}$$

More Linearization Solution

$$\frac{\partial V}{\partial x} = 0$$

$$\frac{\partial V}{\partial y_1} = \left(\frac{\partial V}{\partial y}\right)\left(\frac{\partial y}{\partial y_1}\right) + \left(\frac{\partial V}{\partial y_1}\right)$$

$$= \frac{mg}{2} + ky_1$$

$$\frac{\partial V}{\partial y_2} = \left(\frac{\partial V}{\partial y}\right)\left(\frac{\partial y}{\partial y_2}\right) + \left(\frac{\partial V}{\partial y_2}\right)$$

$$= \frac{mg}{2} + ky_2$$

More Linearization. Solution
Put it all together

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_1}\right) = \frac{m(\ddot{y}_1 + \ddot{y}_2)}{2} - \frac{J}{2L}\frac{(\ddot{y}_2 - \ddot{y}_1)}{1 - \left(\frac{y_2 - y_1}{2L}\right)^2}$$

$$-\frac{2J}{(2L)^2}\frac{(\dot{y}_2 - \dot{y}_1)^2}{\left[1 - \left(\frac{y_2 - y_1}{2L}\right)^2\right]^2} = -\frac{mg}{2} - ky_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_2}\right) = \frac{m(\ddot{y}_1 + \ddot{y}_2)}{2} + \frac{J}{2L}\frac{(\ddot{y}_2 - \ddot{y}_1)}{1 - \left(\frac{y_2 - y_1}{2L}\right)^2}$$

$$+\frac{2J}{(2L)^2}\frac{(\dot{y}_2 - \dot{y}_1)^2}{\left[1 - \left(\frac{y_2 - y_1}{2L}\right)^2\right]^2} = -\frac{mg}{2} - ky_2$$

More Linearization. Solution
Linearize

$$\frac{m(\ddot{y}_1 + \ddot{y}_2)}{2} - \frac{J}{2L}(\ddot{y}_2 - \ddot{y}_1) = -\frac{mg}{2} - ky_1$$

and

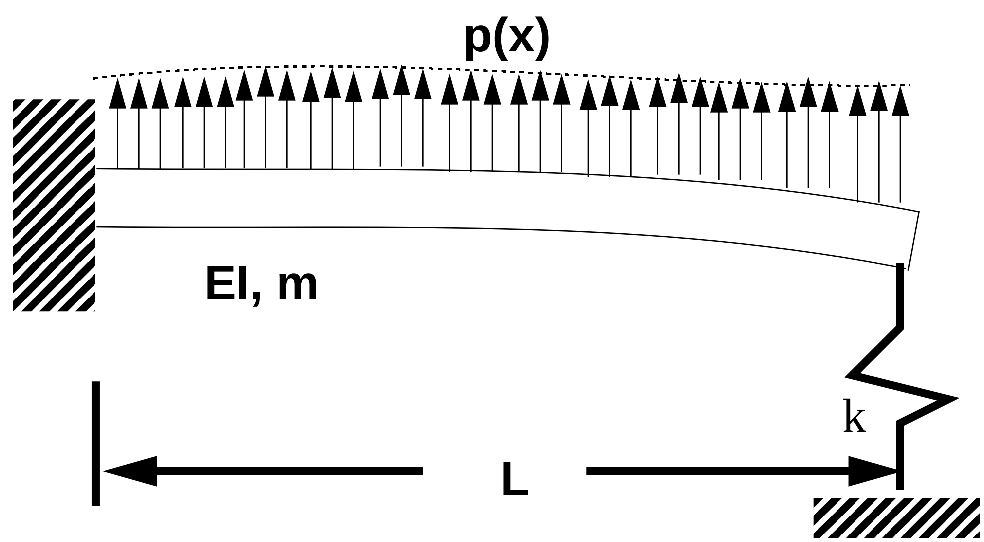
$$\frac{m(\ddot{y}_1 + \ddot{y}_2)}{2} + \frac{J}{2L}(\ddot{y}_2 - \ddot{y}_1) = -\frac{mg}{2} - ky_2$$

Please verify the above.

Though it is common to start linearizing earlier in the process, premature linearization can sometimes lead to error. We shall see this later.

Another Example of Lagrange Equations With Assumed Modes

Consider the cantilevered beam shown at right. The beam has bending modulus EI, and a spring of stiffness k at it end. There is a distributed load p(x,t) imposed as shown.



We postulate a defomation espressed as

$$y(x,t) = A_1(t)\frac{x^2}{L^2} + A_2(t)\frac{x^2(L-x)}{L^3}$$

$$= A_1(t)f_1(x) + A_2(t)f_2(x)$$

We shall derive equations for the evolution of $A_1(t)$ and $A_2(t)$. Note that, by construction, $A_1(t)$ and $A_2(t)$ have dimensions of length.

More Lagrange Equations With Distributed Displacement

Lets calculate the Kinetic Energy:

$$T(A_1, A_2) = \int_0^L \frac{m}{2}(\dot{y})^2 dx$$

$$= \frac{1}{2}[A_1(t)^2 I_1 + 2A_1(t)A_2(t)I_2 + A_2(t)I_3]$$

where

$$I_1 = \int_0^L m(f_1(x))^2 dx = \frac{mL}{5}$$

$$I_2 = \int_0^L m f_1(x) f_2(x) dx = \frac{mL}{30}$$ **and**

$$I_3 = \int_0^L m(f_2(x))^2 dx = \frac{mL}{105}$$

More Lagrange Equations With Distributed Displacement

Lets calculate the Potential Energy:

$$V(A_1, A_2) = \int_0^L \frac{EI}{2}(y'')^2 dx + \frac{k}{2}(y(L))^2$$

$$= \frac{1}{2}[A_1(t)^2 I_4 + 2A_1(t)A_2(t)I_5 + A_2(t)I_6]$$

where

$$I_4 = \int_0^L EI(f_1''(x))^2 dx + k(f_1(L))^2$$

$$= 4EI/L^3 + k$$

,

$$I_5 = \int_0^L EI f_1''(x) f_2''(x) dx + k f_1(L) f_2(L)$$

, and

$$= -2EI/L^3$$

$$I_6 = \int_0^L EI(f_2''(x))^2 dx + k(f_2(L))^2 = 4EI/L^3$$

More Lagrange Equations With Distributed Displacement

Lets finally calculate the generalized forces due to the applied pressure.

The virtual work associated with a virtual displacement δA_1 is

$$\delta W_1 = \int_0^L p(x,t)\left(\frac{\partial y}{\partial A_1}\right)\delta A_1 dx = \int_0^L p(x,t) f_1(x) dx \delta A_1$$

=> $F_1(t) = \delta W_1 / \delta A_1 = \int_0^L p(x,t) f_1(x) dx$

Similarly

$$F_2(t) = \delta W_2 / \delta A_2 = \int_0^L p(x,t) f_2(x) dx$$

More Lagrange Equations With Distributed Displacement

Put it all together to find

$$\begin{bmatrix} I_1 & I_2 \\ I_2 & I_3 \end{bmatrix} \begin{bmatrix} \ddot{A}_1 \\ \ddot{A}_2 \end{bmatrix} + \begin{bmatrix} I_4 & I_5 \\ I_5 & I_6 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

Plugging in integrated values:

$$\imath L \begin{bmatrix} \frac{1}{5} & \frac{1}{30} \\ \frac{1}{30} & \frac{1}{105} \end{bmatrix} \begin{bmatrix} \ddot{A}_1 \\ \ddot{A}_2 \end{bmatrix} + \frac{EI}{L^3} \begin{bmatrix} \left(4 + \frac{kL^3}{EI}\right) & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

Next Time

Review Of Homework From Lecture 8

A Short Quiz On Linearization.

Linearization Of Lagrange Equations

Properties Of Resulting Matrix Equations