Lecture 17
Rods and More Strings

These are 1-dimensional structures.

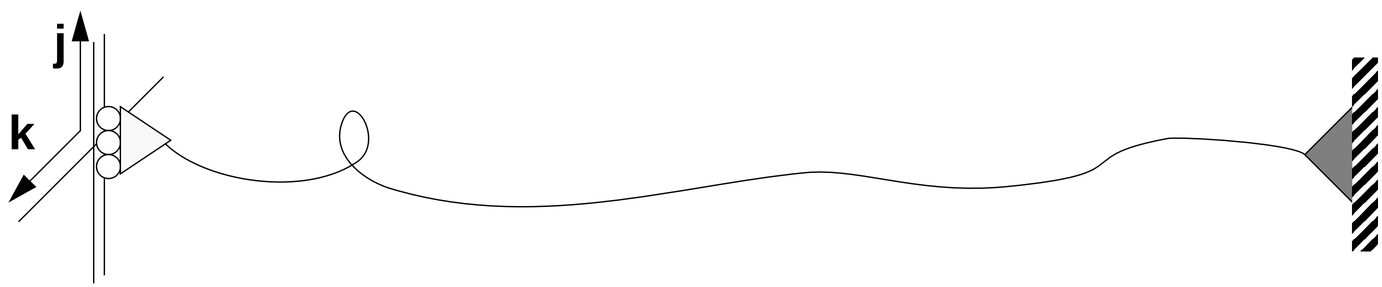
- **With rods, only longitudinal deformation is considered.**
- **With strings, only lateral motions were considered.**

A More Complicated String Problem
Deformations in Two Dimensions

We studied the string vibrating in a plane earlier. Lets now consider a string that can move in either of two lateral directions.

$$\vec{y}(x, t) = u(x, t)\vec{j} + v(x, t)\vec{k}$$

Further, we consider a more complicated boundary condition.



We restrict the left hand end of the string to move within a track in the vertical direction: $\vec{y}(0, t) = s(t)\vec{j}$ **(=>** $v(0, t) = 0$**).**

The string is pinned on the right-hand-side:

$\vec{y}(L, t) = 0$ **(=>** $u(L, t) = v(L, t) = 0$**).**

A More Complicated String Problem Deformations in Two Dimensions

We also consider a distributed load $\vec{p}(x, t)$.

Again, we employ Hamilton's principle. We shall need each of the terms

in $0 = \int_{t_1}^{t_2} (\delta T - \delta V + \delta W) dt$**.**

$T = \frac{1}{2}\int_0^L \rho \dot{\vec{y}}(x, t) \bullet \dot{\vec{y}}(x, t) dt$, $V = \frac{1}{2}\int_0^L F \vec{y}'(x, t) \bullet \vec{y}'(x, t) dt$**, and**

$$\delta W = \int_0^L \vec{p}(x, t) \bullet \overrightarrow{\delta y}(x, t) dt$$

A More Complicated String Problem
Deformations in Two Dimensions

$$\int_{t_1}^{t_2} \delta T\, dt = \int_{t_1}^{t_2} \left(\int_0^L \rho \dot{\vec{y}}(x,t) \bullet \overrightarrow{\delta y}(x,t)\, dx \right) dt$$

$$= \int_0^L \left(\int_{t_1}^{t_2} \left[\frac{\partial}{\partial t}(\rho \dot{\vec{y}} \bullet \overrightarrow{\delta y}) - \rho \ddot{\vec{y}} \bullet \overrightarrow{\delta y} \right] dt \right) dx$$

$$= \int_0^L \underbrace{(\rho \dot{\vec{y}} \bullet \overrightarrow{\delta y}) \Big|_{t_1}^{t_2}}_{0} dx - \int_0^L \left(\int_{t_1}^{t_2} \rho \ddot{\vec{y}} \bullet \overrightarrow{\delta y}\, dt \right) dx$$

A More Complicated String Problem
Deformations in Two Dimensions

$$\int_{t_1}^{t_2} \delta V\, dt = \int_{t_1}^{t_2} \left(\int_0^L F \vec{y}'(x,t) \bullet \overrightarrow{\delta y}'(x,t)\, dx \right) dt$$

$$= \int_{t_1}^{t_2} \left(\int_0^L F \left[\frac{\partial}{\partial x} (\vec{y}' \bullet \overrightarrow{\delta y}) - F \vec{y}'' \bullet \overrightarrow{\delta y} \right] dt \right) dx$$

$$= \int_{t_1}^{t_2} (F \vec{y}' \bullet \overrightarrow{\delta y}) \Big|_0^L dt - \int_0^L \left(\int_{t_1}^{t_2} F \vec{y}'' \bullet \overrightarrow{\delta y}\, dt \right) dx$$

A More Complicated String Problem
Deformations in Two Dimensions

We put this all together to find: $\int_0^L \left(\int_{t_1}^{t_2} [F\ddot{}\vec{y}'' - \rho\ddot{\vec{y}} + \vec{p}] \bullet \overrightarrow{\delta y}\, dt \right) dx = 0$

for all $\delta\vec{y}$ **from which we deduce that** $\rho\ddot{\vec{y}} = F\vec{y}'' + \vec{p}$ **<=>**

$\rho\ddot{u} = Fu'' + \vec{p} \bullet \vec{j}$ **and** $\rho\ddot{v} = Fv'' + \vec{p} \bullet \vec{k}$

Also, $\int_{t_1}^{t_2} (F\vec{y}' \bullet \overrightarrow{\delta y})\Big|_0^L dt = 0$ **from which we conclude that**

$\int_{t_1}^{t_2} F\vec{y}'(0,t) \bullet \overrightarrow{\delta y}(0,t)\, dt = 0$ **=>** $\vec{y}'(0,t) \bullet (\delta s \vec{j})$ **=>** $u'(0,t) = 0$

A More Complicated String Problem
Deformations in Two Dimensions

We have two independent partial differential equations with their own boundary conditions:

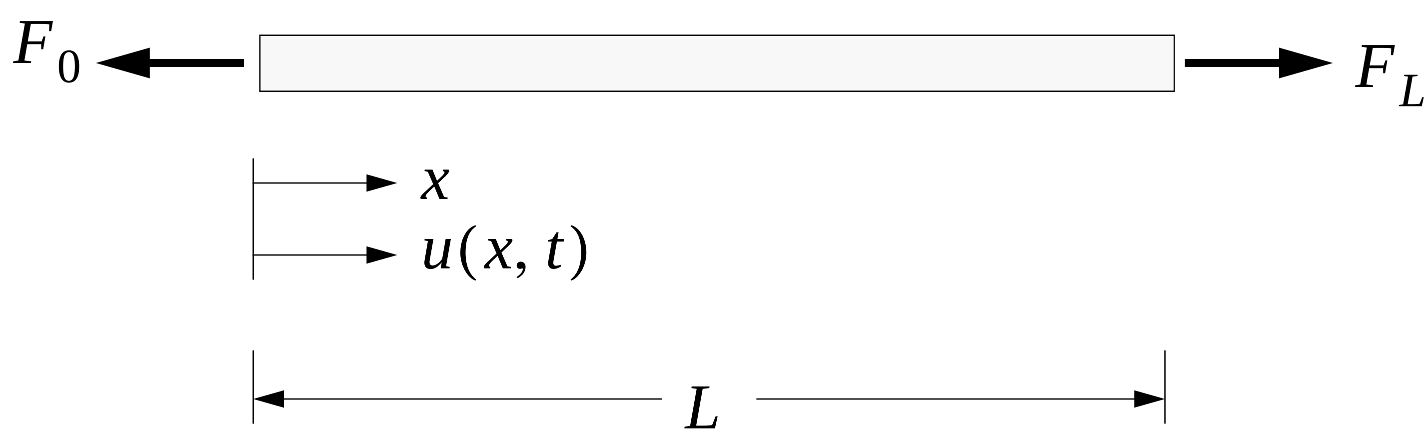
$$\rho \ddot{u} = F u'' + \vec{p}(x, t) \bullet \vec{j} \text{ subject to } u'(0, t) = 0 \text{ \& } u(L, t) = 0$$

and

$$\rho \ddot{v} = F v'' + \vec{p}(x, t) \bullet \vec{k} \text{ subject to } v(0, t) = 0 \text{ \& } v(L, t) = 0$$

We shall discuss how to solve these equations later in this lecture.

Governing Equations for Rods By Hamilton's Principle



We consider a rod of length L, cross sectional area A, density ρ, and Young's modulus E. We wish to derive a governing equation for this.

We shall use Hamilton's Principle.

Governing Equations for Rods By Hamilton's Principle

Recall: The Weak form of Hamilton's Principle: asserts that the actual path is one about which $\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\,dt = 0$**. Here,** δW **is the virtual work of external forces pushing the system from one path in configuration space to another.**

Lets enumerate the components of the above equation:

$$V = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx \qquad , T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx,$$

and

$$\delta W = -F_0\,\delta u(0, t) + F_L\,\delta u(L, t)$$

Governing Equations for Rods By Hamilton's Principle

For the rod, Hamilton's principle becomes

$$0 = \int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\,dt$$

$$= \int_{t_1}^{t_2}\left(\int_0^L \rho A \dot{u}\,\delta\dot{u}\,dx\right)dt - \int_{t_1}^{t_2}\left(\int_0^L EA u'\,\delta u'\,dx\right)dt$$

$$+ \int_{t_1}^{t_2} (-F_0(t)\,\delta u(0,t) + F_L(t)\,\delta u(L,t))\,dt$$

$$= \int_0^L\int_{t_1}^{t_2} \rho A\left(\frac{\partial}{\partial t}[\dot{u}\,\delta u] - \ddot{u}\,\delta u\right)dt\,dx - \int_{t_1}^{t_2}\int_0^L EA\left(\frac{\partial}{\partial x}[u'\,\delta u] - u''\,\delta u\right)dx\,dt$$

$$+ \int_{t_1}^{t_2} (-F_0(t)\,\delta u(0,t) + F_L(t)\,\delta u(L,t))\,dt$$

Governing Equations for Rods
By Hamilton's Principle

After some re-organization, this becomes

$$0 = \int_0^L (\rho A \dot{u} \delta u)\Big|_{t_1}^{t_2} dx - \int_0^L \int_{t_1}^{t_2} (\rho A \ddot{u} - EAu'')\delta u \, dt \, dx$$

$$- \int_{t_1}^{t_2} (EAu' \delta u)\Big|_0^L dt + \int_{t_1}^{t_2} [-F_0(t)\delta u(0, t) + F_L(t)\delta u(L, t)] dt$$

0

From which we conclude that $\rho A \ddot{u} - EAu''$**, that**

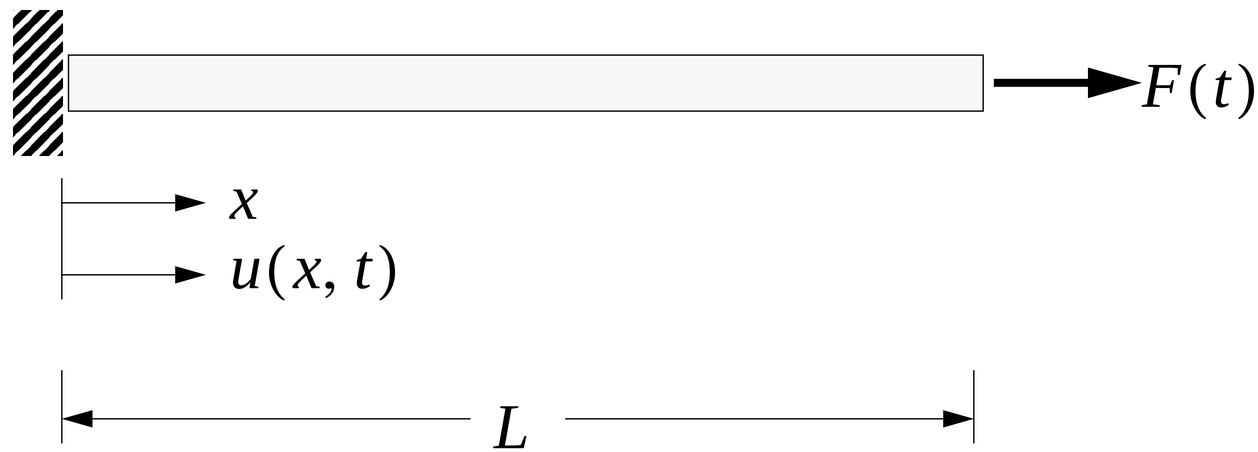
$[EAu'(L, t) - F_L(t)]\delta u(L, t) = 0$ **and that**

$[EAu'(0, t) - F_0(t)]\delta u(0, t) = 0$**.**

Hamilton's principle provides governing equations and form of B.C.

Solutions for Rods

Say we have a specific problem. Our rod is fixed on one end and subject to a prescribed load on the other.



We shall try to solve this using a modal solution, but first, we must find the modes.

We consider the *unforced* beam and solve $\rho A\ddot{u} - EAu'' = 0$ **subject to the boundary conditions:** $u(0, t) = 0$ **&** $u'(L, t) = 0$.

From where does the second B.C. come?

Modal Solutions for Rods

We postulate solutions $u(x, t) = Re(Ce^{i\omega t})X(x)$. This yields the equation for X: $X'' + \lambda^2 X = 0$ where $\lambda^2 = \frac{\omega^2\rho}{E}$.

The only solutions to the above equation with the specified boundary conditions are $X_n(x) = \alpha_n \sin\left(\frac{\pi(2n-1)}{2}\frac{x}{L}\right)$.

Note that, as expected, $(X_m, MX_n) = 0$ for $m \neq n$.

Modal Solutions for Rods
Mass Normalization

Lets choose the constants α_n **so that the modes are orthonormal with respect to the mass matrix.** $(X_m, MX_n) = \delta_{mn}$ **requires that**

$$\alpha_n = \sqrt{\frac{2}{L\rho}} \text{ so that } X_n(x) = \sqrt{\frac{2}{L\rho}}\sin\left(\frac{\pi(2n-1)}{2}\frac{x}{L}\right).$$

Please confirm that this normalization is correct.

Modal Solutions for Rods

Lets now make a change of variables to discretize the problem.

Assume that $u(x, t) = \sum_{n=1}^{N} f_n(t) X_n(x)$

Let now derive our governing equations in terms of our new variables $f_n(t)$ **using Lagrange equations. (Lagrange has been a good friend to us.)**

$$T = \frac{1}{2}\int_0^L \rho A \left(\sum_{m=1}^{N} \dot{f}_m(t) X_m(x) \right) \left(\sum_{n=1}^{N} \dot{f}_n(t) X_n(x) \right) dx$$

$$= \frac{1}{2}\left(\sum_{m=1}^{N} \sum_{n=1}^{N} \dot{f}_m(t) \dot{f}_n(t) \int_0^L \rho A X_m(x) X_n(x) dx \right) = \frac{1}{2} \sum_{n=1}^{N} \dot{f}_n(t)^2$$

Modal Solutions for Rods

The potential energy is a little more complicated:

$$V = \frac{1}{2}\int_0^L EA\left(\sum_{m=1}^{N} f_m(t)X_m'(x)\right)\left(\sum_{n=1}^{N} f_n(t)X_n'(x)\right)dx$$

$$= \frac{1}{2}\left(\sum_{m=1}^{N}\sum_{n=1}^{N} f_m(t)f_n(t)\int_0^L EAX_m'(x)X_n'(x)dx\right)$$

$$= \frac{1}{2}\left(\frac{\pi^2 EA}{\rho L^2}\right)\sum_{n=1}^{N} n^2 f_n(t)^2$$

Modal Solutions for Rods

The generalized force associated with each mode calculated from the work done by the physical forces against each mode:

$$\delta W_n = \delta f_n \; X_n(L)F(t) = \delta f_n \; Q_n^X(t) \Rightarrow Q^X{}_n(t) = X_n(L)F(t)$$

The constituents of the Lagrange equation are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{f}_n}\right) - \frac{\partial T}{\partial f_n} = \ddot{f}_n, -\frac{\partial V}{\partial f_n} = -\left(\frac{\pi^2 EA}{\rho L^2}\right) n^2 f_n$$

The Lagrange equations for this system are:

$$\ddot{f}_n = -\left(\frac{\pi^2 EA}{\rho L^2}\right) n^2 f_n + F(t)\sqrt{\frac{2}{L\rho}}\sin\left(\frac{\pi(2n-1)}{2}\right)$$

Modal Solutions for Rods

The forcing term simplifies a little bit:

$$\ddot{f}_n = -\left(\frac{\pi^2 EA}{\rho L^2}\right) n^2 f_n - F(t)\sqrt{\frac{2}{L\rho}}(-1)^n$$

We might solve this by Laplace transform methods or we might solve this by numerical integration.

Having found a sufficient number of the $f_n(t)$, we can reconstitute the physical solution from $u(x, t) = \sum_{n=1}^{N} f_n(t) X_n(x)$.

Modal Solutions for Rods

To re-cap:

- **Having identified the potiential and kinetic energy, we can use Hamilton's principle to deduce the governing equations.**
- **Modal solutions are found to the unforced problem**
- **Mode shapes are used to discretize the problem and can be used to introduce generalized degrees of freedom in solving the forced vibration problem.**
- **Using modal coordinates diagonalizes the problem for us.**

Please think about how we could introduce damping into this analysis.

Next Time

Numerical Solutions for Strings and Rods
Galerkin Method

We saw how we could use the eigen functions to convert the partial differential equations into systems of ordinary differential equations that could be solved by