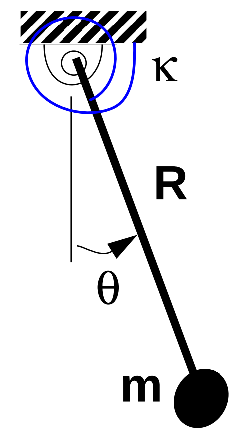
Slides of Lecture 10

Today's Class:

Review of Homework from Lecture 8
A short quiz on linearization.
Linearization of Lagrange Equations
Properties of Resulting Matrix Equations

Homework from Lecture 8
A numerical experiment with linearization

Many times we have derived the equations for a spring reinforced pendulum: $\ddot{\theta} + \left(\frac{\kappa}{mR^2}\theta + \frac{g}{R}\sin\theta \right) = 0$.



The linearized form is $\ddot{\theta} + \omega_L^2\theta = 0$ where

$$\omega_L^2 = \frac{\kappa}{mR^2} + \frac{g}{R}.$$

We use the linearized frequency to dimensionless the time parameter.

Define $\tau = \omega_L t$, define $\phi(\tau) = \theta(\tau/\omega_L)$, and define $\alpha = \frac{g/R}{\omega_L^2}$

Homework for Lecture 8 continued

Then $\frac{d^2\phi}{d\tau^2} = \frac{d^2\theta}{dt^2}\frac{1}{\omega_L^2}$ and $\frac{d^2\phi}{d\tau^2} + [(1-\alpha)\phi + \alpha\sin\phi] = 0$

1. Solve numerically the dimensionless governing equation for the initial conditions: $\phi(0) = \pi$ and $\left.\frac{d\phi}{d\tau}\right|_0 = 0$ over the period $(0, 6\pi)$ for the three cases: $\alpha = 0$, $\alpha = 1/2$, and $\alpha = 1$.

2. Do the same as above but for the initial conditions $\phi(0) = \frac{\pi}{6}$ and

$$\left.\frac{d\phi}{d\tau}\right|_0 = 0$$

3. Compare and discuss the your results for parts 1 and 2.

Solution
Using Matlab

Using state space formulation:

$$x = \begin{bmatrix} \dot{\phi} \\ \phi \end{bmatrix} \text{, then } \dot{x} = \begin{bmatrix} -\alpha \sin(x_2) - (1-\alpha)x_2 \\ x_1 \end{bmatrix}$$

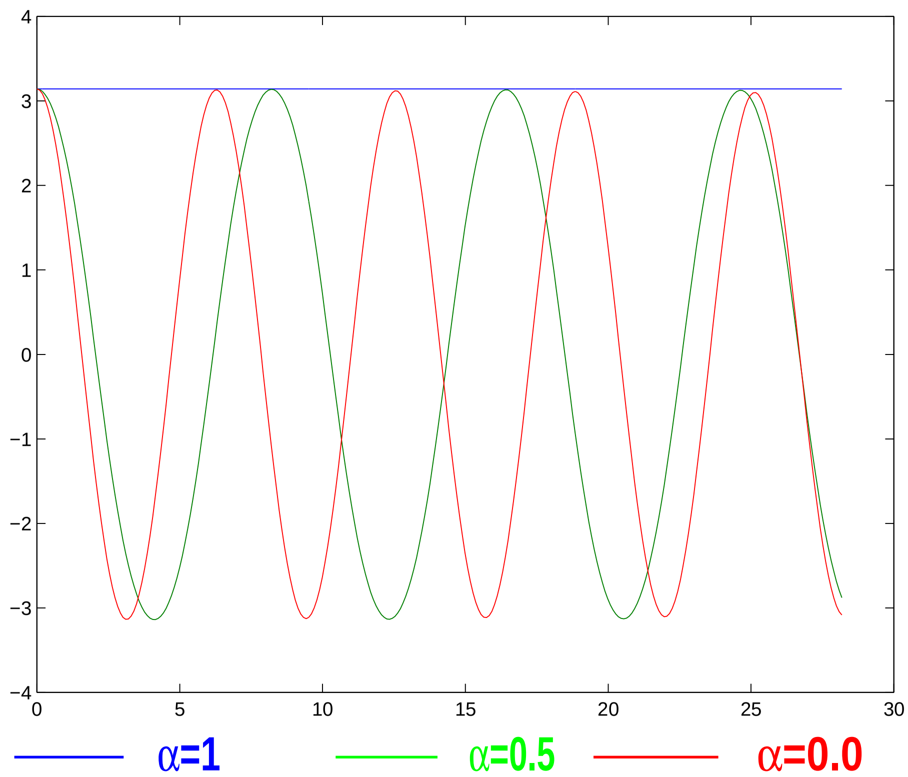
This appears as a function 'pendu1.m' provided to matlab:

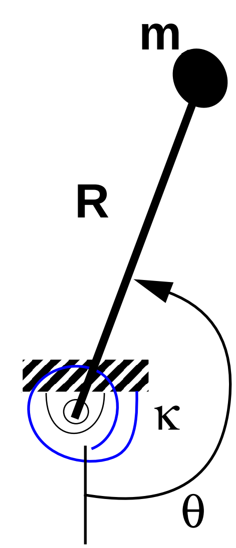
```
function yprime = pendu1(t,x)
global alpha;
yprime = [ -alpha.*sin(x(2)) - (1-alpha).*x(2);  x(1)];
return;
```

Results
Calls to Matlab


```
T = [0:299]*(6*pi)/200;                    %define time array
%
 phi_max = pi;                             % Initial Condition φ=π
%
 alpha = 1;
 [t,x] = ode23('pendu1', T , [0 phi_max]); %Runge-Kuta Integration
 x1 = x;
%
 alpha = 0.5;
 [t,x] = ode23('pendu1', T , [0 phi_max]);
 x2 = x;
%
 alpha = 0.0;
 [t,x] = ode23('pendu1', T , [0 phi_max]);
 x3 = x;
%
 plot(t,x1(:,2), t,x2(:,2), t,x3(:,2));
 print pi -depsc
%
%
 phi_max = pi/6;
%
 alpha = 1;
 [t,x] = ode23('pendu1', T , [0 phi_max]);
 x1 = x;
%
 alpha = 0.5;
 [t,x] = ode23('pendu1', T , [0 phi_max]);
 x2 = x;
%
 alpha = 0.0;
 [t,x] = ode23('pendu1', T , [0 phi_max]);
 x3 = x;
%
 plot(t,x1(:,2), t,x2(:,2), t,x3(:,2));
 print pi6 -depsc
```

Solution for $\phi_0=\pi$

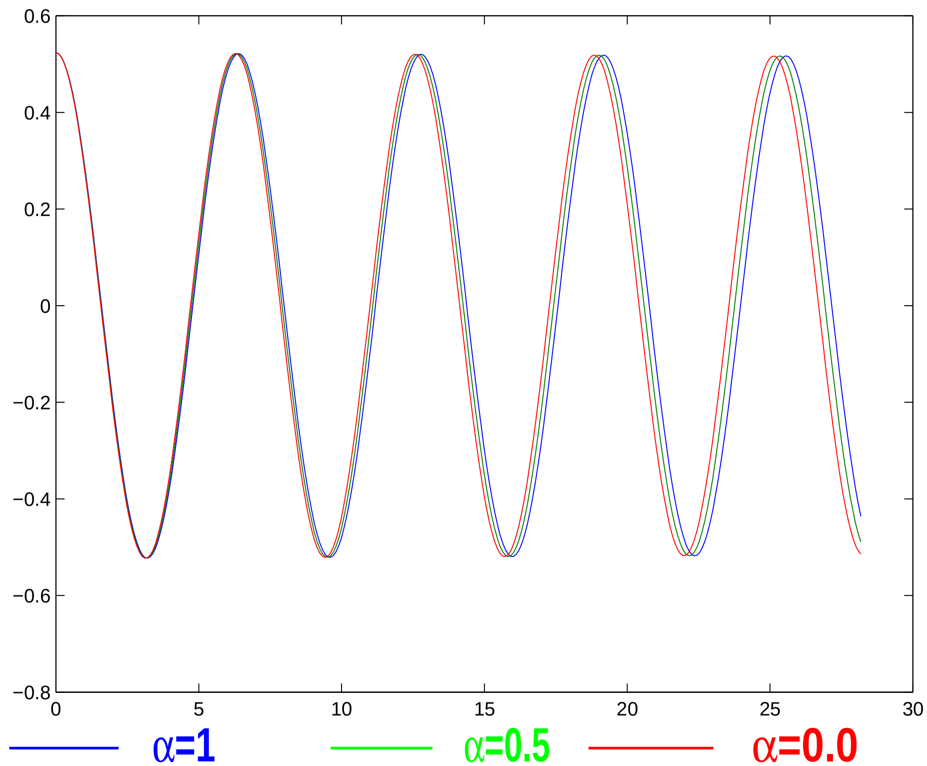


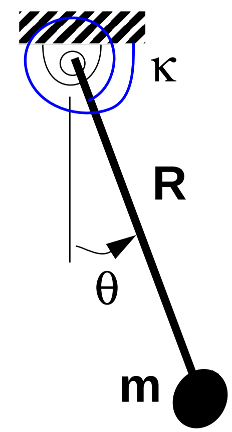


When the pendulum starts displaced nearly vertically, the restoring force due to gravity is also near zero.

The more of the restoring force that is due to the torsional string, the more that the response will appear to be harmonic.

Solution for $\phi_0=\pi/6$





When the pendulum starts only a little distance from the static equilibrium, the ‘small angle’ approximation is very good.

In that case, the linearity of the result is nearly independent of whether the restoring force is due to the spring or to gravity.

Legitimate linearization requires that the linearized terms be nearly equal to the corresponding nonlinear terms throughout the deformation.

Short Quiz

This quiz should take about 15 minutes.

Formal Linearization of Lagrange Equations
Derivation of System Matrices

In the limiting cases of small deflection ($|q_r| \ll 1$), small rates $|\dot{q}_r| \ll 1$, the Lagrange equations simplify.

We begin by looking at kinetic energy.

$$T = \sum_n \frac{m_n}{2} \dot{\vec{x}}_n \cdot \dot{\vec{x}}_n = \sum_n \frac{m_n}{2} \sum_r \sum_s \left(\frac{\partial \dot{\vec{x}}_n}{\partial \dot{q}_r} \cdot \frac{\partial \dot{\vec{x}}_n}{\partial \dot{q}_s} \right) \dot{q}_r \dot{q}_s$$

Again, we make use of one of the core observations in the derivation of the Lagrange equations: $\dfrac{\partial \dot{\vec{x}}_n}{\partial \dot{q}_r} = \dfrac{\partial \vec{x}_n}{\partial q_r}$

$$T = \sum_n \frac{m_n}{2} \sum_r \sum_s \left(\frac{\partial \vec{x}_n}{\partial q_r} \cdot \frac{\partial \vec{x}_n}{\partial q_s} \right) \dot{q}_r \dot{q}_s = \sum_r \sum_s \frac{1}{2} \dot{q}_r \dot{q}_s \hat{M}_{rs}(\{q\})$$

Formal Linearization of Lagrange Equations Derivation of System Matrices

In the limit of small displacement about the configuration of stable equilibrium $\{q^s\}$, the mass matrix becomes

$$M_{rs} = \hat{M}_{rs}(\{q^s\}) = \sum_n \frac{m_n}{2}\left(\left(\frac{\partial \vec{x}_n}{\partial q_r}\right)\bigg|_{\{q^s\}} \cdot \left(\frac{\partial \vec{x}_n}{\partial q_s}\right)\bigg|_{\{q^s\}}\right).$$

Note that $M_{rs} = \hat{M}_{rs}(\{q^s\})$ is still symmetric.

Important: In the limit of small displacement,

$$T = \sum_r \sum_s \frac{1}{2}\dot{q}_r \dot{q}_s M_{rs} \text{ \& } M_{rs} = \frac{\partial^2 T}{\partial \dot{q}_r \partial \dot{q}_s}$$

Formal Linearization of Lagrange Equations
Derivation of System Matrices

Observe that the mass matrix is positive semi-definite. Consider an array of generalized speeds $\{\dot{q}^t\}$. The corresponding kinetic energy is

$$
\begin{aligned}
T(\{\dot{q}^t\}) &= \sum_{r,s} \frac{1}{2} M_{rs} \dot{q}_r^t \dot{q}_s^t \\
&= \sum_{r,s} \frac{1}{2} \dot{q}_r^t \dot{q}_s^t \sum_n \frac{m_n}{2} \left(\left(\frac{\partial \vec{x}_n}{\partial q_r} \right) \Bigg|_{\{q^s\}} \cdot \left(\frac{\partial \vec{x}_n}{\partial q_s} \right) \Bigg|_{\{q^s\}} \right) \\
&= \sum_n \frac{m_n}{2} \sum_r \sum_s \left(\left(\frac{\partial \vec{x}_n}{\partial q_r} \right) \Bigg|_{\{q^s\}} \cdot \left(\frac{\partial \vec{x}_n}{\partial q_s} \right) \Bigg|_{\{q^s\}} \right) \dot{q}_r^t \dot{q}_s^t \\
&= \sum_n \frac{m_n}{2} \dot{\vec{y}}_n^t \cdot \dot{\vec{y}}_n^t \geq 0 \text{ where } \dot{\vec{y}}_n^t = \sum_r \left(\frac{\partial \vec{x}_n}{\partial q_r} \right) \Bigg|_{\{q^s\}} \dot{q}_r^t
\end{aligned}
$$

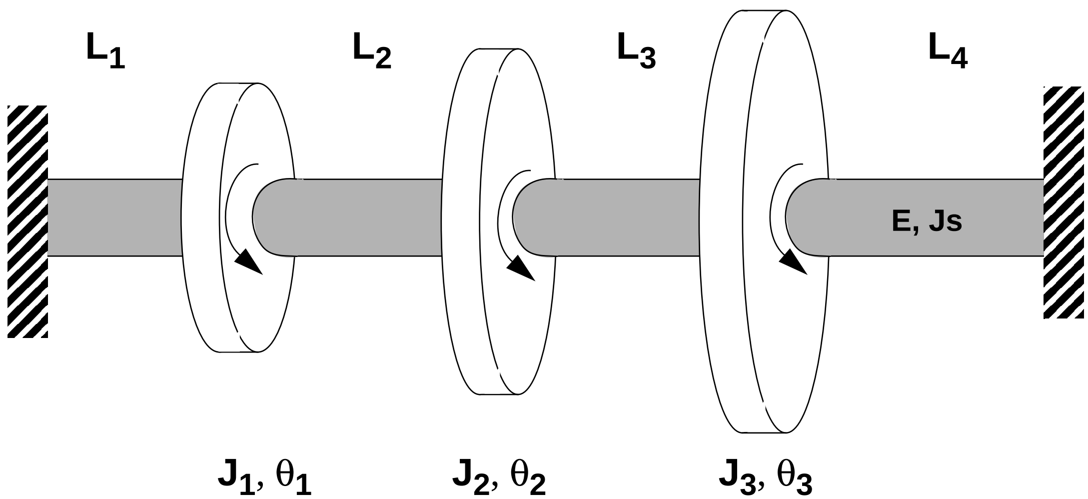
The meaning of the mass matrix.

Consider the "acceleration" force

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_r}\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_r}\frac{1}{2}\sum_s\sum_t M_{st}\dot{q}_s\dot{q}_t\right) = \sum_s M_{rs}\ddot{q}_s.$$

Next, consider column vector of generalized accelerations that are all zeros except for a "1" on the r'th row: $\{\ddot{q}\}_r = \begin{bmatrix} 0 & . & 1 & . & . & 0 \end{bmatrix}^T$**. The resulting "acceleration forces" seen by each of the other generalized degrees of freedom are the r'th column of** M**.**

Formal Linearization of Lagrange Equations
Example of Mass Matrix



Lets consider a structure similar to the disk-shaft system in Meirovitch.

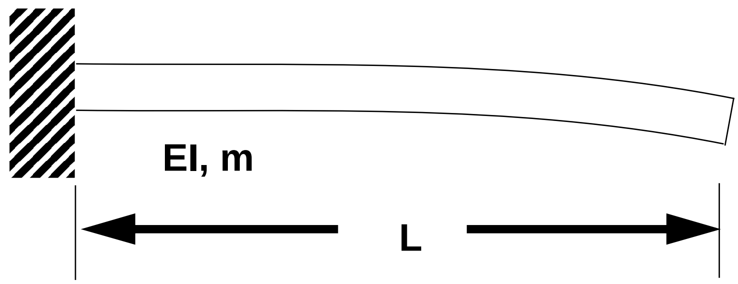
Ignoring the kinetic energy of the shaft, the kinetic energy of the systems is

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}J_3\dot{\theta}_3^2.$$

Our mass matrix is $[M_{ij}] = \left[\frac{\partial^2 T}{\partial\dot{\theta}_i\partial\dot{\theta}_j}\right] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$

Another
Example of Mass Matrix

Consider a cantilevered beam for which we postulate a displacement field with two free generalized coordinates:



$$y(x,t) = A_1(t)f_1(x) + A_2(t)f_2(x)$$

The kinetic energy in the beam will be

$$T(A_1, A_2) = \int_0^L \frac{m}{2}(\dot{y})^2 dx$$

$$= \frac{1}{2}[\dot{A}_1(t)^2 I_1 + 2\dot{A}_1(t)\dot{A}_2(t)I_2 + (2\dot{A}_2(t))I_3]$$

where $I_1 = \int_0^L m(f_1(x))^2 dx$, $I_3 = \int_0^L m(f_2(x))^2 dx$

and $I_2 = \int_0^L m f_1(x) f_2(x) dx$

Another Example of Mass Matrix Continued

Our mass matrix is $[M_{ij}] = \left[\dfrac{\partial^2 T}{\partial \dot{A}_i \partial \dot{A}_j}\right] = m\begin{bmatrix} I_1 & I_2 \\ I_2 & I_3 \end{bmatrix}$

Formal Linearization of Lagrange Equations
Derivation of System Matrices

Consider the potential energy in the vicinity of a configuration of stable equilibrium. In an equilibrium configuration, the sum of the potential forces in each direction is zero $0 = -\frac{\partial V}{\partial q_r}$**.**

Taylor series expansion for V **is**

$$V(\{q\}) = V_0 + \frac{1}{2}\sum_r \sum_s \left.\frac{\partial^2 V}{\partial q_r \partial q_s}\right|_{\{q^s\}} \Delta q_r \Delta q_s + \mathrm{H.O.T.}.$$

We define the stiffness matrix to be $K_{rs} = \left.\frac{\partial^2 V}{\partial q_r \partial q_s}\right|_{\{q^s\}}$ **.**

Note that, by construction, K **is symmetric.**

Formal Linearization of Lagrange Equations
Derivation of System Matrices

Potential energy in terms of Stiffness. In the limit of small strains,

$$V = V_0 + \frac{1}{2}\sum_r \sum_s K_{rs} q_r q_s$$

where we have redefined the generalized coordinates to be zero at the equilibrium configuration. Usually, we set the datum so that $V_0 = 0$.

Because stability requires that potential energy be at a local minimum, $\sum_r \sum_s K_{rs} q_r q_s \geq 0$ for all $\{q\}$. This is an assertion that the stiffness matrix is positive semi-definite.

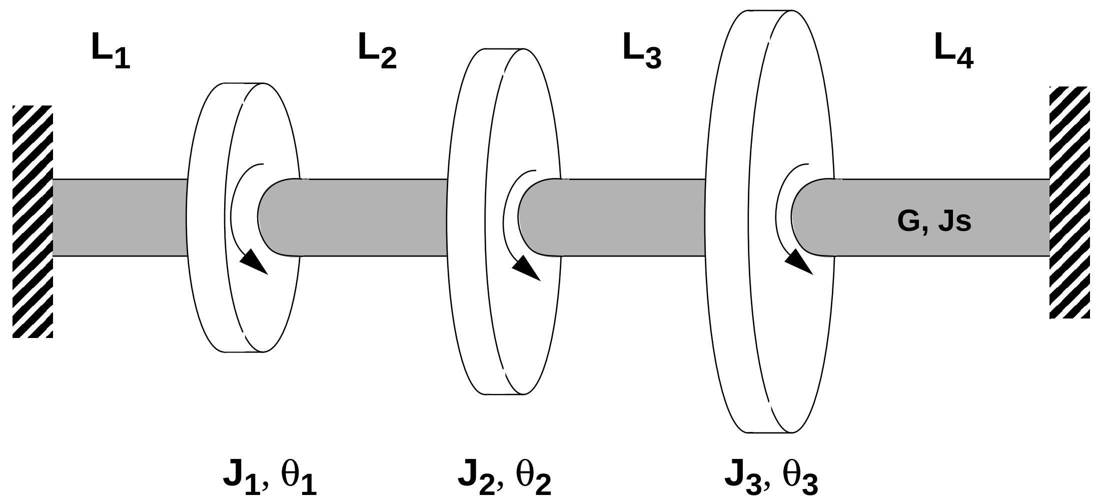
The meaning of the Stiffness matrix.

Consider the generalized force

$$F_r = -\frac{\partial}{\partial q_r}\left(\frac{1}{2}\sum_s\sum_t K_{st} q_s q_t\right) = -\sum_s K_{rs} q_s .$$

Next, consider column vector of generalized displacement that are all zeros except for a "1" on the r'th row: $\{q\}_r = \begin{bmatrix} 0 & . & 1 & . & . & 0 \end{bmatrix}^T$**. The resulting force seen by each of the other generalized degrees of freedom are the r'th column of** K**.**

An Example Problem of Involving the Stiffness Matrix



Lets consider a structure similar to the disk-shaft system in Meirovitch.

The strain energy in the shaft is computed in terms of the differences in the rotations at the disks

$$V = \frac{1}{2} G J_s \left[\frac{\theta_1^2}{L_1} + \frac{(\theta_2 - \theta_1)^2}{L_2} + \frac{(\theta_3 - \theta_2)^2}{L_3} + \frac{\theta_3^2}{L_4} \right].$$

An Example Problem of Involving the Stiffness Matrix

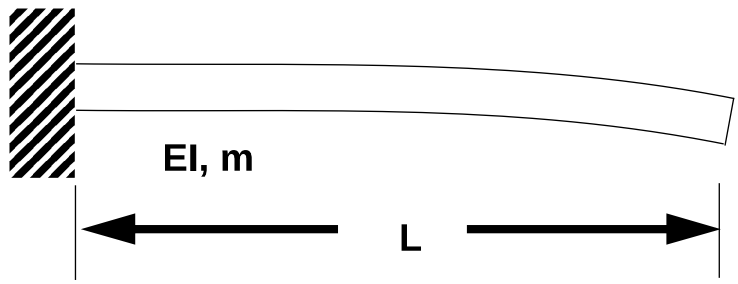
Our stiffness matrix is

$$[K_{ij}] = \left[\frac{\partial^2 V}{\partial\theta_i \partial\theta_j}\right] = GJ_s \begin{bmatrix} \frac{1}{L_1} + \frac{1}{L_2} & -\frac{1}{L_2} & 0 \\ -\frac{1}{L_2} & \frac{1}{L_2} + \frac{1}{L_3} & -\frac{1}{L_3} \\ 0 & -\frac{1}{L_3} & \frac{1}{L_3} + \frac{1}{L_4} \end{bmatrix}$$

Note that the i, jth element of K is the torque felt on disk i due to a unit rotation imposed on disk j.

Another Example Involving the Stiffness Matrix

Consider a cantilevered beam for which we postulate a displacement field with two free generalized coordinates:



$$y(x, t) = A_1(t) f_1(x) + A_2(t) f_2(x)$$

The strain energy in the beam will be

$$V(A_1, A_2) = \int_0^L \frac{EI}{2}(y'')^2 dx$$

$$= \frac{1}{2}[A_1(t)^2 I_4 + 2A_1(t)A_2(t)I_5 + A_2(t)^2 I_6]$$

$$I_4 = \int_0^L EI(f_1''(x))^2 dx, I_6 = \int_0^L EI(f_2''(x))^2 dx$$

and $I_5 = \int_0^L m f_1''(x) f_2''(x) dx$

Another Example of Mass Matrix Continued

Our stiffness matrix is $[K_{ij}] = \left[\frac{\partial^2 V}{\partial A_i \partial A_j}\right] = EI\begin{bmatrix} I_4 & I_5 \\ I_5 & I_6 \end{bmatrix}$

Matrix Multiplication
A short review

Say a and b are column vectors of length N, then $b^T a = a^T b$ are defined to be $\sum_{r=1}^{N} a_r b_r$. This is the inner product of algebraic vectors.

Say A is an N by N matrix, then the matrix vector product Aa is defined so that the rth element of Aa is $\left\{ \sum_{s=1}^{N} A_{rs} a_s \right\}$

Combining the above two definitions: $a^T A b = \left(\sum_{r=1}^{N} \sum_{s=1}^{N} A_{rs} a_r b_s \right)$

Matrix Multiplication
A short review

The product of two matrices is defined in a similar way. Say B is also an N by N matrix also, then the product AB is defined as

$$(AB)_{rs} = \sum_{t=1}^{N} A_{rt} B_{ts}$$

The transpose of a matrix is that which is obtained by reversing the order of the indices: $(A^T)_{rs} = (A)_{sr}$. The transpose of products is found to be $((AB)^T)_{rs} = (AB)_{sr} = \sum_{t=1}^{N} A_{st} B_{tr} = (B^T A^T)_{rs}$. In matrix notion: $(AB)^T = B^T A^T$

Flexibility Matrix
(Lets start using matrix notation)

In those cases where there are no rigid body motions, the stiffness matrix is nonsingular, and we can define a flexibility matrix

$$a = K^{-1}.$$

In this case we observe that since $F = -Kq$, $q = -aF$.

The strain energy may now be expressed in terms of force

$$V = \frac{1}{2}q^T K q = \frac{1}{2}(-aF)^T K(-aF) = \frac{1}{2}F^T aF$$

Sometimes vibration problems are formulated in terms of the flexibility matrix, though the use of displacement-based finite elements makes flexibility formulations decreasingly popular.

Discussion of the flexibility matrix is presented here for completeness. We shall do little more with it in this class.

Matrix Transformations

Say that we have derived the following matrix representation of a linear system: $M\ddot{q} + Kq = F$ where the column vector F contains all externally applied load.

Now, say we select another set of generalized coordinates with which we can specify the first set: $q = T\beta$. Lets re-derive our governing equations in terms of β.

Kinetic Energy:

$$T = \frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2}(T\dot{\beta})^T M (T\dot{\beta}) = \frac{1}{2}\dot{\beta}^T (T^T M T)\dot{\beta}$$

Strain Energy: $V = \frac{1}{2}q^T K q = \frac{1}{2}(T\beta)^T K (T\beta) = \frac{1}{2}\beta^T (T^T K T)\beta$

Potential Energy of Loading:

$$A = -q^T F = -(T\beta)^T F = -\beta^T (T^T F)$$

Matrix Transformation
Lagrange Equation in New System

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}_r}\right) - \cancelto{0}{\frac{\partial T}{\partial \beta_r}} + \frac{\partial V}{\partial \beta_r} = -\frac{\partial A}{\partial \beta_r}$$

becomes $\tilde{M}\ddot{\beta} + \tilde{K}\beta = \tilde{F}$

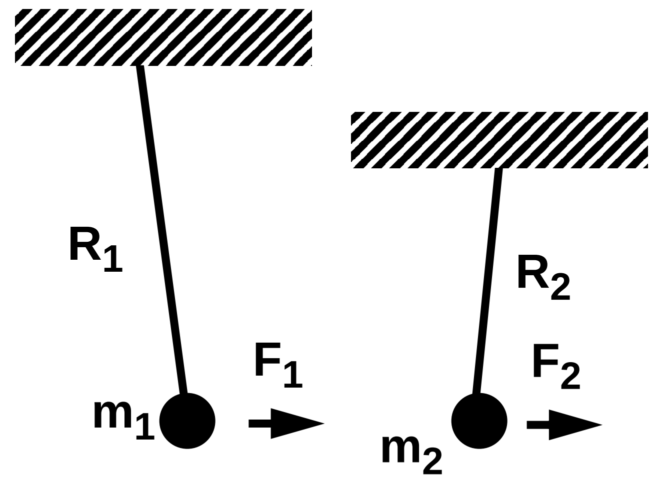
where $\tilde{M} = T^T M T$**,** $\tilde{K} = T^T K T$**, and** $\tilde{F} = T^T F$

Note that the mass and stiffness matrices remain symmetric and at least positive semi-definite.

If T **is a nonsingular, square matrix, the transformation** $T^T A T$ **is called a congruence transformation of A.**

Matrix Transformation

We often use such transformations when introducing constraints to a system. Here we consider two independent pendulums. The kinetic and potential energies are



$$T = \frac{m_1}{2}R_1^2\dot{\theta}_1^2 + \frac{m_2}{2}R_2^2\dot{\theta}_2^2 \text{ and}$$

$$V = -mgR_1\cos\theta_1 - mgR_2\cos(\theta_2)$$

The potential energy of loading is

$$A = -F_1R_1\theta_1 - F_2R_2\theta_2$$

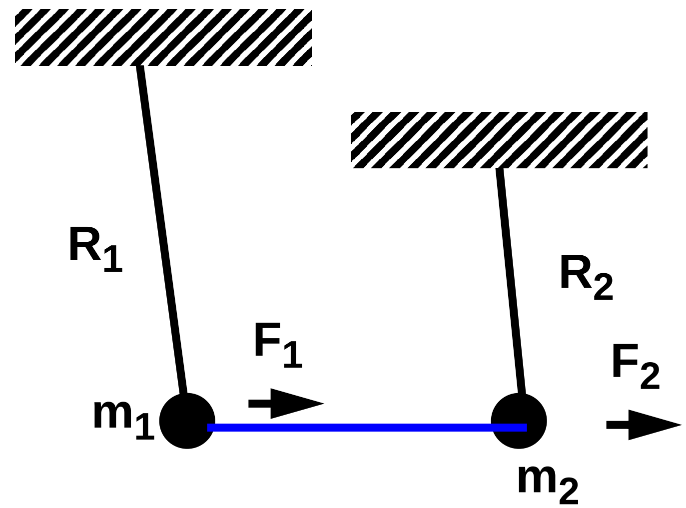
The linearized equations of motion are

$$\begin{bmatrix} m_1R_1^2 & 0 \\ 0 & m_2R_2^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} mgR_1 & 0 \\ 0 & mgR_2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F_1R_1 \\ F_2R_2 \end{bmatrix}$$

Matrix Transformation
Lets add a constraint

Consider the effect of connecting the two masses with a massless rod. This has the effect of requiring

$R_1\theta_1 = R_2\theta_2$, **permitting us to write**



$$\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}1\\ R_1/R_2\end{bmatrix}\theta_1.$$

Define $T = \begin{bmatrix}1\\ R_1/R_2\end{bmatrix}$**. Then**

$$\tilde{M} = \begin{bmatrix}1 & R_1/R_2\end{bmatrix}\begin{bmatrix}m_1R_1^2 & 0\\ 0 & m_2R_2^2\end{bmatrix}\begin{bmatrix}1\\ R_1/R_2\end{bmatrix} = \begin{bmatrix}(m_1 + m2)R_1^2\end{bmatrix}$$

Matrix Transformation
With a constraint

The stiffness matrix is

$$\tilde{K} = \begin{bmatrix} 1 & R_1/R_2 \end{bmatrix} \begin{bmatrix} m_1 g R_1 & 0 \\ 0 & m_2 g R_2 \end{bmatrix} \begin{bmatrix} 1 \\ R_1/R_2 \end{bmatrix}$$

$$= \begin{bmatrix} m_1 g R_1 + m_2 g \dfrac{R_1^2}{R_2} \end{bmatrix}$$

and the applied force is

$$\tilde{F} = \begin{bmatrix} 1 & R_1/R_2 \end{bmatrix} \begin{bmatrix} F_1 R_1 \\ F_2 R_2 \end{bmatrix} = \begin{bmatrix} R_1(F_1 + F_2) \end{bmatrix}$$

Matrix Transformation With a constraint

The resulting equation for θ_1 is

$$[(m_1 + m2)R_1^2]\ddot{\theta}_1 + \left(m_1 g R_1 + m_2 g \frac{R_1^2}{R_2}\right)\theta_1 = R_1(F_1 + F_2)$$

The addition of a constraint reduced the number of active degrees of freedom by one.

Next Time

Short Orientation

LAB TOUR