Lecture 19: Membranes
Formulations And Some Analytic Solutions

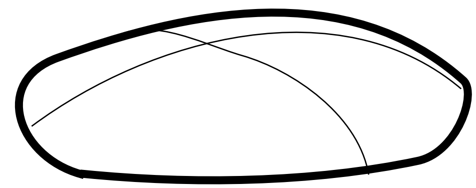
Membranes are two-dimensional systems which, at rest, will occupy the configuration of minimum curvature consistent with the boundary conditions. The restoring force of these membranes derives from an isotropic surface tension.

In what follows, we assume that the boundaries lie entirely in a plane. In this case, the membrane will also lie in a plane when at rest.

There are some cases for which it is possible to achieve closed form solutions for the vibration of membranes. The best known of these are those with rectangular or circular boundaries.

Governing Equations
Hamilton's Principle Again

The only part of this problem which is not immediately obvious is the calculation of potential energy. We identify the strain energy with the surface tension and the surface area of the membrane. The strain and kinetic energies and virtual work are:



$$V = \iint_A \tau \sqrt{1 + \nabla w \bullet \nabla w}\, dA \qquad T = \frac{1}{2}\iint_A \rho \dot{w}^2 dA$$

$$\approx \iint_A \tau \left[1 + \frac{1}{2} \nabla w \bullet \nabla w \right] dA \qquad \delta W = \iint_A p \delta W dA$$

where w is the out-of-plane displacement of the membrane, F is the surface tension in the membrane, ρ is the mass per unit area of the membrane, and integration is over the area enclosed by the boundaries.

Governing Equations
Hamilton's Principle Again

$$0 = \int_{t_1}^{t_2}\iint_A \rho \dot{w}\frac{\partial}{\partial t}\delta w dA - \int_{t_1}^{t_2}\iint \tau \nabla w \bullet \nabla \delta w dA + \int_{t_1}^{t_2}\iint_A p \delta W dA$$

As usual, we must do an integration by parts with respect to time to move the $\frac{\partial}{\partial t}\delta w$ **into** δw **and we must do an integration by parts (Gauss's theorem) with respect to the spacial coordinate to move the** $\nabla \delta w$ **into** δw**.**

$$0 = \int_{t_1}^{t_2}\iint_A \rho\left[\frac{\partial}{\partial t}(\dot{w}\delta w) - \ddot{w}\delta w\right] dA$$

$$-\int_{t_1}^{t_2}\iint_A \tau[\nabla \bullet (\nabla w \delta w) - \nabla^2 w \delta w] dA$$

$$+\int_{t_1}^{t_2}\iint_A p \delta W dA$$

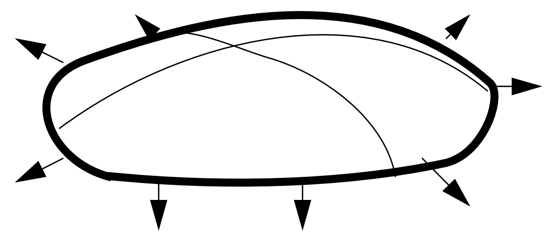
Governing Equations
Hamilton's Principle Again

Evaluating the above and grouping like terms,

$$0 = \int_{t_1}^{t_2} \iint_A [\rho \ddot{w} - \tau \nabla^2 w - p] \delta w \, dA$$ **for all δw on A subject to**

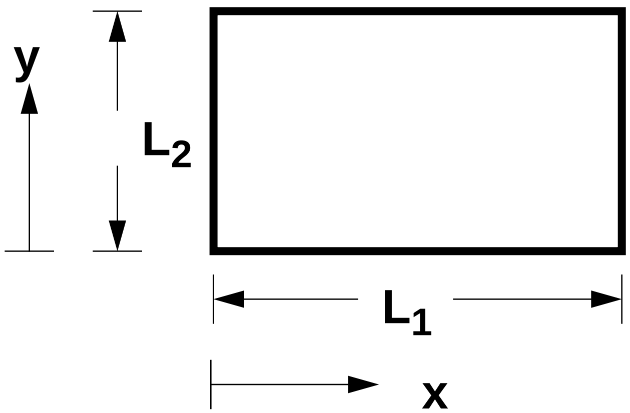
$\delta w = 0$ on ∂A. From the above, $\rho \ddot{w} - \tau \nabla^2 w - p = 0$ on the interior of A.

Also, $\int_{t_1}^{t_2} \int_{\partial A} \tau [\nabla w \bullet \vec{n}] \delta w \, d\Gamma = 0$ where the integral is over the boundary ∂A of A.



Here, $\vec{n}$ is a unit outward normal. This asserts that at locations on the boundary for which w is not fixed, $\nabla w \bullet \vec{n} = 0$. In the following, we will consider only those case for which w is fixed on the boundaries.

Membrane Solutions on a Rectangle



In rectangular coordinates,

$$\tau\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)w + p(x, y, t) = \rho\ddot{w}$$

. Boundary conditions are

$w(0, y, t) = w(L_1, y, t) = 0$ **and**

$w(x, 0, t) = w(x, L_2, t) = 0$**.**

Lets first try solving the homogeneous equation via separation of variables (implying eigen analysis).

$$w(x, y, t) = \sum_{\lambda} f_{\lambda}(t) W_{\lambda}(x, y)$$

This leads to $\dfrac{\nabla^2 W_{\lambda}}{W_{\lambda}} = \dfrac{\rho}{\tau}\dfrac{\ddot{f}_{\lambda}}{f_{\lambda}} = -\lambda^2$

Rectangular Membrane Eigen Analysis

$\nabla^2 W_\lambda + \lambda^2 W_\lambda = 0$ **is satisfied by** $W_\lambda = \sin\frac{\pi m x}{L_1}\sin\frac{\pi n y}{L_2}$

We see that this works if $\lambda^2 = \left(\frac{\pi m}{L_1}\right)^2 + \left(\frac{\pi n}{L_2}\right)^2$**. (we could have derived the above by doing another separtation of variables:**
$W_\lambda(x, y) = \sum X_\lambda(x) Y_\lambda(y)$**.)**

Substituting back into our governing equation, we see that our eigen solutions, $W_\lambda = \sin\frac{\pi m x}{L_1}\sin\frac{\pi n y}{L_2}$**, have frequency** $\omega_\lambda = \sqrt{\frac{\tau}{\rho}}\lambda$

We shall exploit the mutual orthogonality of the eigenmodes. As usual, we ortho*normalize* the modes: $(\rho W_{\lambda_1}, W_{\lambda_2}) = \delta_{\lambda_1, \lambda_2}$**.**

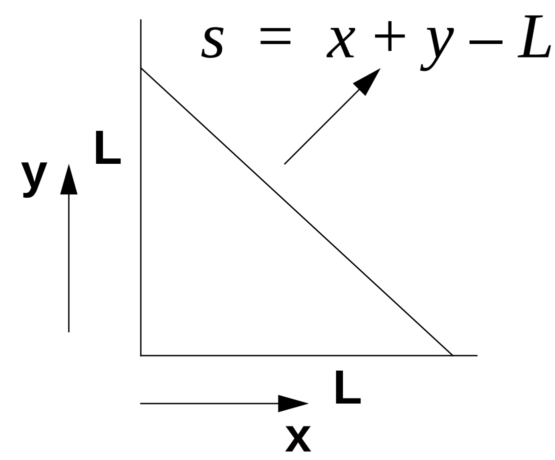
Forced Oscillation of Rectangular Membranes

The governing equation is $\rho\ddot{w} - \tau\nabla^2 w - p(x, y) = 0$. Just as we have done in the past, we express $w(x, y, t) = \sum_{\lambda} f_{\lambda}(t) W_{\lambda}(x, y)$, and derive Lagrange equations using the $f_{\lambda}(t)$ as generalized coordinates. After exploitation of the orthonormality, we have independent equations for the modal coefficients:

$$\ddot{f}_{\lambda} + \frac{\tau}{\rho}\lambda^2 f = (p(x, y, t), W_{\lambda})$$

Eigenmodes on triangles
This would be a good assignment

Analytic solutions on other polygons are harder. Lets consider the eigen problem on the right isosceles triangle shown.



We might approximate the deformed shape of the membrane on the triangle by the following polynomials:

$$W = xy(x + y - L)[A_1 + A_2 x + A_3 y]$$

$$= A_1 P_1(x, y) + A_2 P_2(x, y) + A_3 P_3(x, y)$$

We could use these basis functions to approximate the eigenmodes of a membrane on this triangle.

Approximate Eigenmodes on triangles

$$W = A_1 P_1(x, y) + A_2 P_2(x, y) + A_3 P_3(x, y)$$

Lets set up the formulation to estimate the eigen shapes:

- **The Lagrange equations are** $[M]\{\ddot{A}\} + [K]\{A\} = 0$ **where** $M_{mn} = \iint \rho P_m P_n dA$ **and** $K_{mn} = \iint \tau \nabla P_m \bullet \nabla P_n dA$
- **Postulate a harmonic motion** $\{A\} = Re(Ce^{i\omega t}\{A_0\})$
- **Solve the resulting eigen problem** $(-\omega^2 M + K)\{A_0\} = 0$ **for three arrays** $\{A^1\}, \{A^2\}, \{A^3\}$
- **Construct the eigen shapes from** $W_k(x, y) = A_1^k P_1(x, y) + A_2^k P_2(x, y) + A_3^k P_3(x, y)$

Vibration of Circular Membranes

In a circular domain, $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$

We assume that there are no holes in the membrane, so the boundary conditions are $w(R, \theta, t) = 0$**.**

Lets first try solving the homogeneous equation via separation of variables (implying eigen analysis).

$$w(r, \theta, t) = \sum_{\lambda} f_{\lambda}(t) W_{\lambda}(r, \theta)$$

This leads to $\dfrac{\nabla^2 W_{\lambda}}{W_{\lambda}} = \dfrac{\rho}{\tau}\dfrac{\ddot{f}_{\lambda}}{f_{\lambda}} = -\lambda^2$

(I hope that this looks very familiar to you.)

Vibration of Circular Membranes

$\nabla^2 W_\lambda + \lambda^2 W_\lambda = 0$ **is itself solved via a separation of variables:** $W_\lambda = R_\lambda(r)\Theta_\lambda(\theta)$. **On substitution back into the differential equation, we find**

$$\frac{\frac{\partial^2}{\partial r^2}R_\lambda + \frac{1}{r}\frac{\partial R_\lambda}{\partial r}}{R_\lambda} + \frac{1}{r^2}\frac{\left(\frac{\partial^2 \Theta}{\partial \theta^2}\right)}{\Theta} + \lambda^2 = 0$$

from which we conclude that $\left(\frac{\partial^2 \Theta}{\partial \theta^2}\right) / \Theta = C$ **where** C **is a constant.**

We also conclude that $\left(\frac{\partial^2}{\partial r^2}R_\lambda + \frac{1}{r}\frac{\partial R_\lambda}{\partial r}\right) + \frac{C}{r^2}R_\lambda + \lambda^2 R_\lambda = 0$.

Vibration of Circular Membranes

In order for the solution to make sense, Θ must be periodic in θ, so

$C = -m^2$ **and** $\left(\dfrac{\partial^2 \Theta}{\partial \theta^2}\right) + m^2 \Theta = 0$ **where m is any nonzero integer.**

The solution to this equation, denoted as Θ_m, is

$$\Theta_m(\theta) = C_{1m} \sin(m\theta) + C_{2m} \cos(m\theta).$$

The equation for the radial function is

$\left(\dfrac{\partial^2}{\partial r^2} R_{\lambda, m} + \dfrac{1}{r}\dfrac{\partial R_\lambda}{\partial r}\right) - \dfrac{m^2}{r^2} R_{\lambda, m} + \lambda^2 R_{\lambda, m} = 0$**. This has solutions**

$R_{\lambda, m}(r) = C_{3m} J_m(\lambda r) + C_{4m} Y_m(\lambda r)$ **where J_m is the Bessel function of the first kind and order m and Y_m is the Bessel function of the second kind and order m.**

Vibration of Circular Membranes

Note that Y_m has a singularity about zero. If we have a membrane without holes, then we must set $C_{4m} = 0$.

We still must satisfy the boundary conditions that $W(R, \theta) = 0$. This requires that $J_m(\lambda R) = 0$. This condition is used to find λ.

This offers a large number of solutions. For each order, m, there is a Bessel function of the first kind. That Bessel function has an infinite number of zeros. Lets call the nth zero of the mth Bessel function $x_{m,n}$.

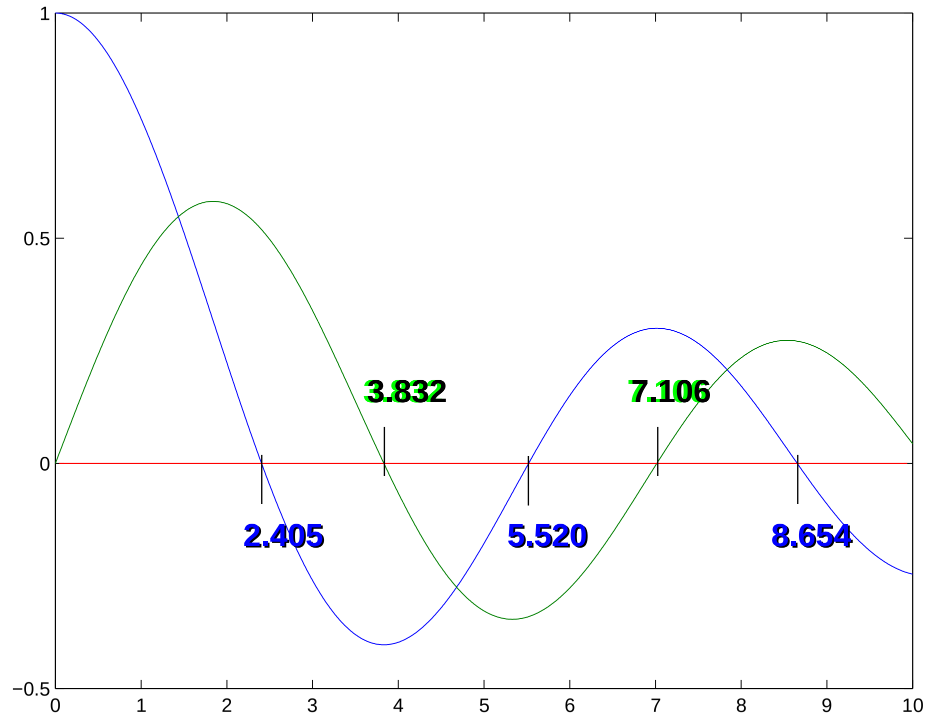
Then $W(r, \theta) = [C^1_{m,n}\sin(m\theta) + C^2_{m,n}\cos(m\theta)]J_m(\lambda_{m,n}r)$ is an eigen mode where $\lambda_{m,n} = x_{m,n}/R$. Frequency is

$$\omega_{m,n} = \sqrt{\tau/\rho}\,\lambda_{m,n}$$

Vibration of Circular Membranes

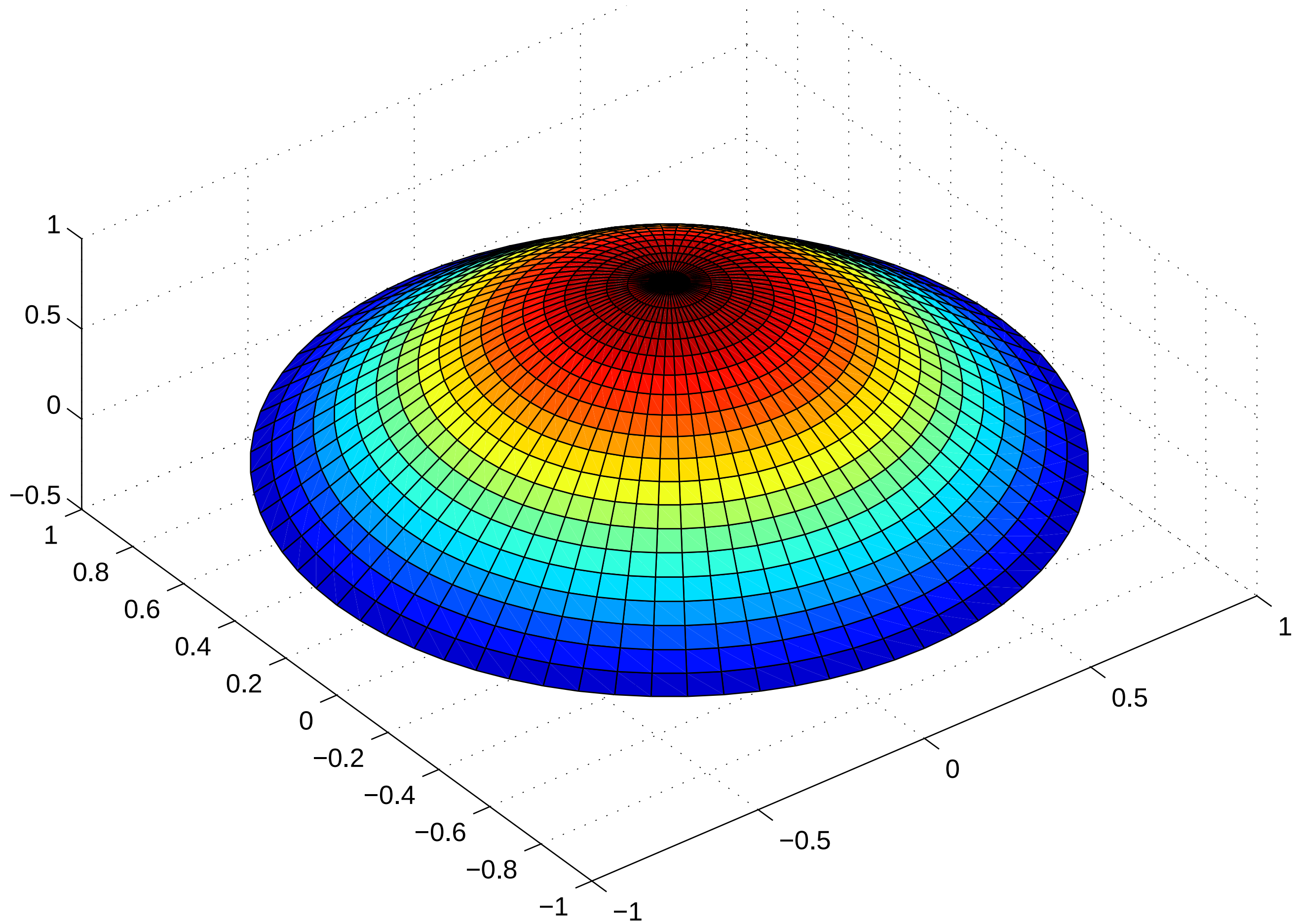
Of course, we can use orthogonality in the usual manner to construct solutions for given initial conditions and loads on the membrane.

It is interesting to see what these modes look like. Following is a plot of the first and second Bessel functions of the first kind

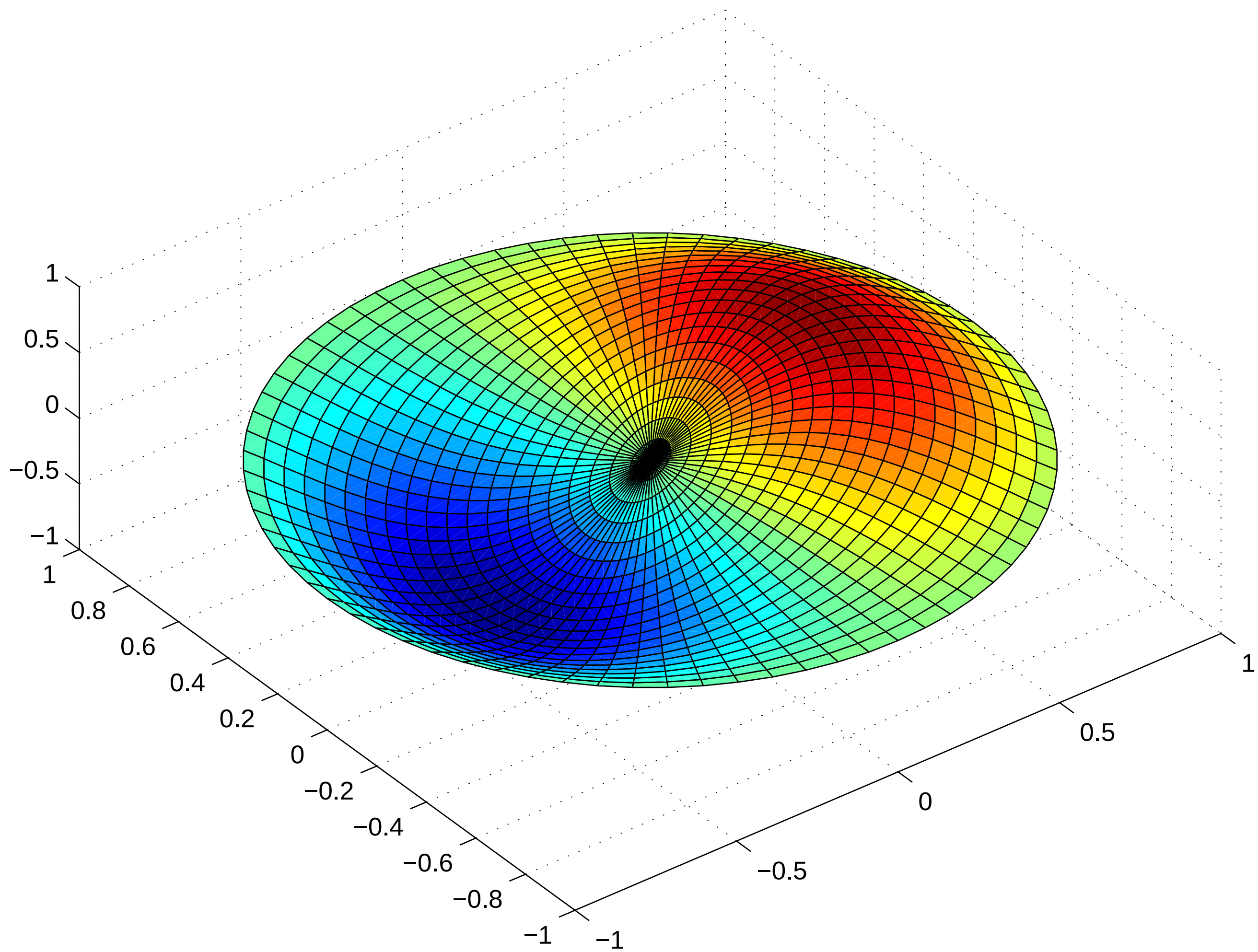


Lets look at the first eigen modes.

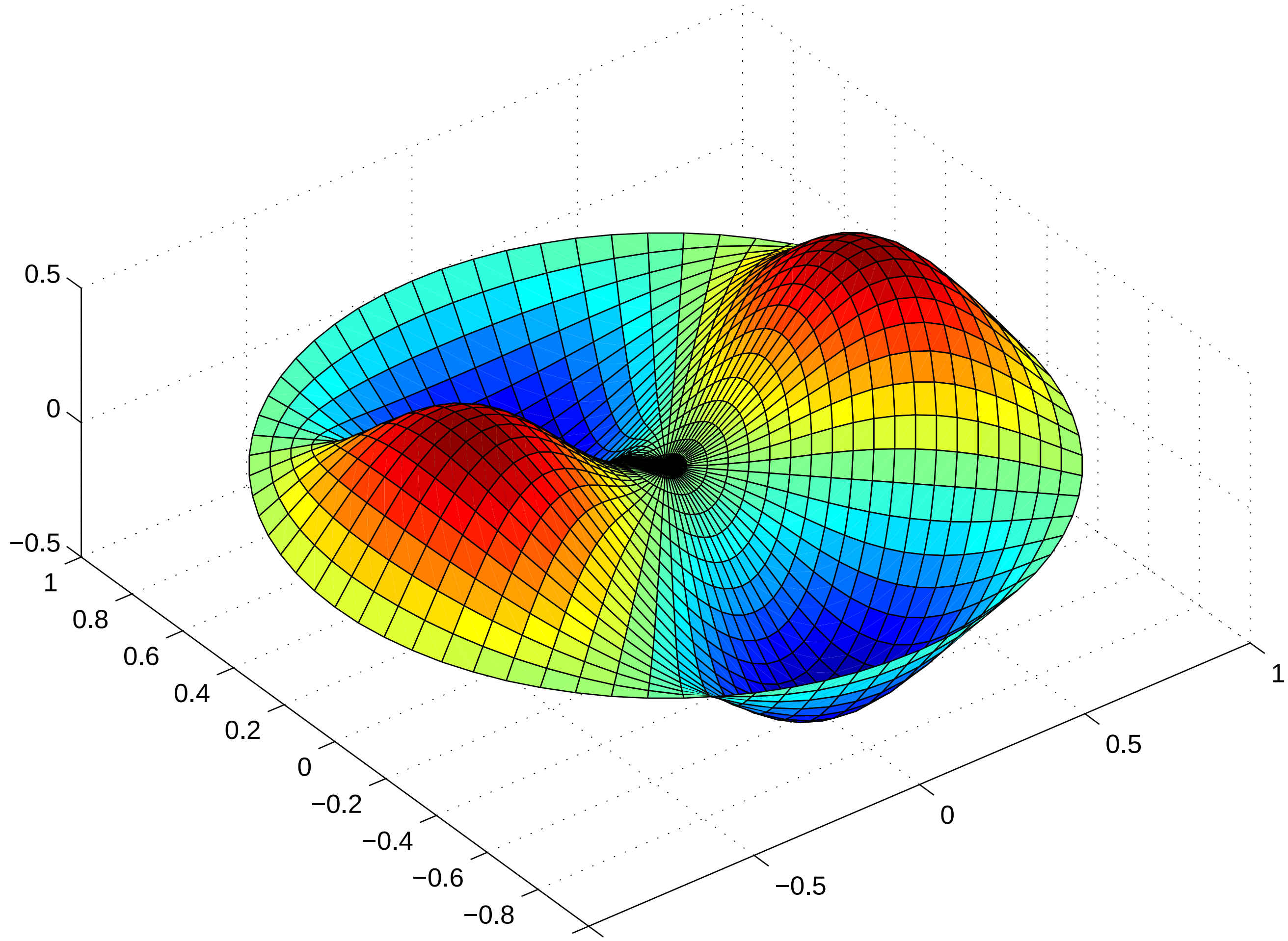
First Eigen Mode
$J_0(2.405r)$ (first zero of J_0)



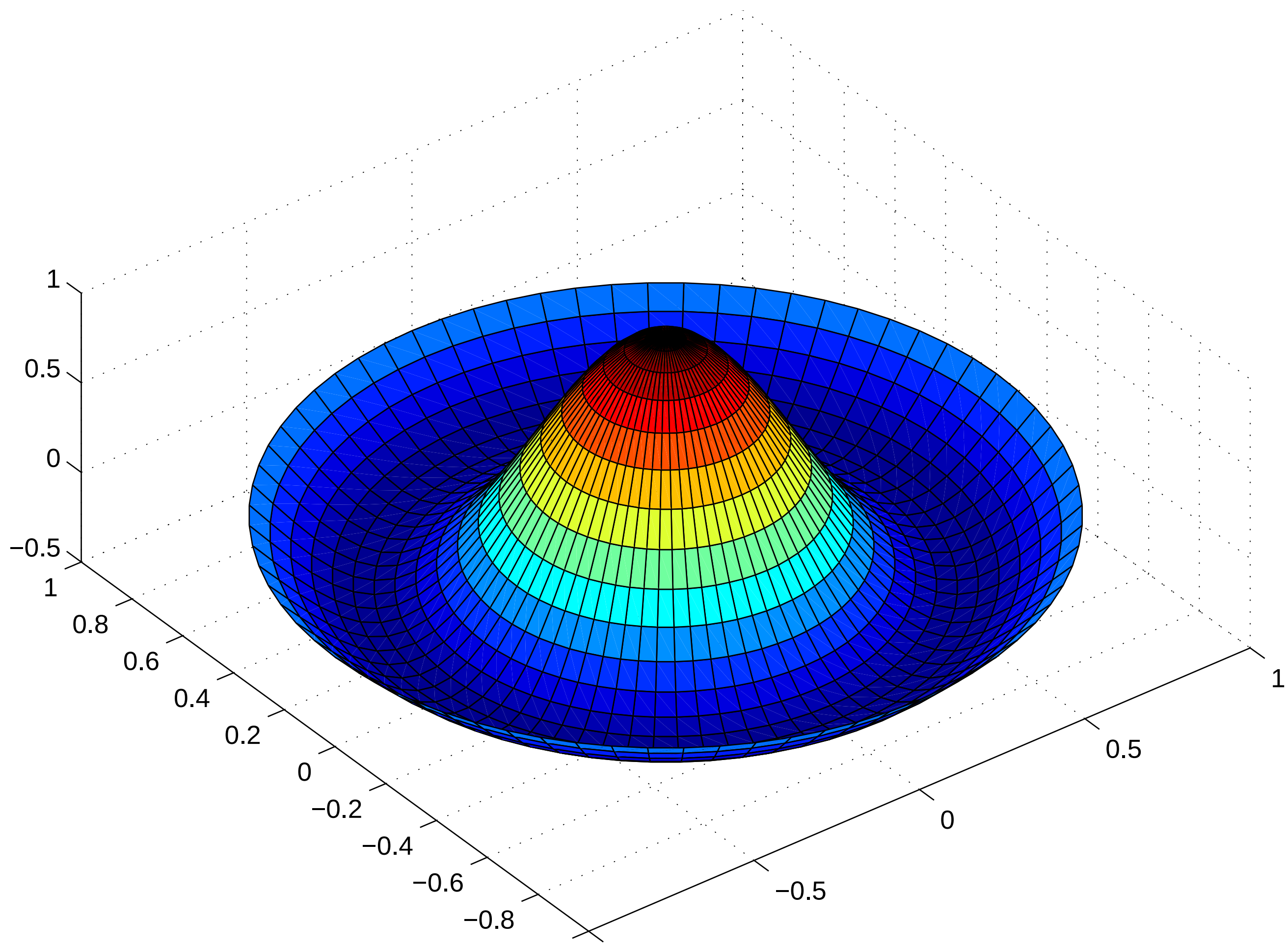
Second Eigen Mode
$J_1(3.832r)$ (first zero of J_1)



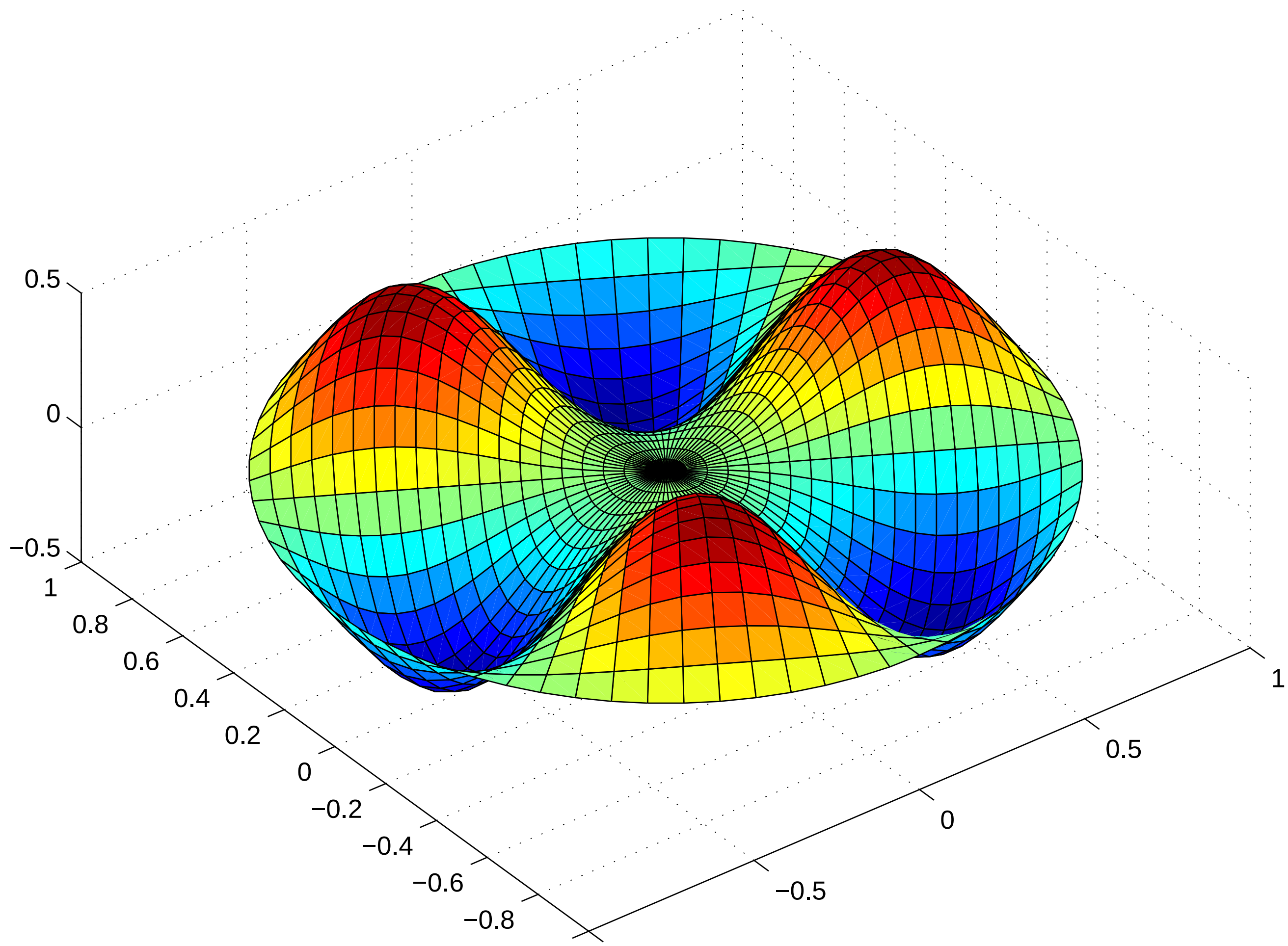
Third Eigen Mode
$J_2(5.136r)$ (first zero of J_2)



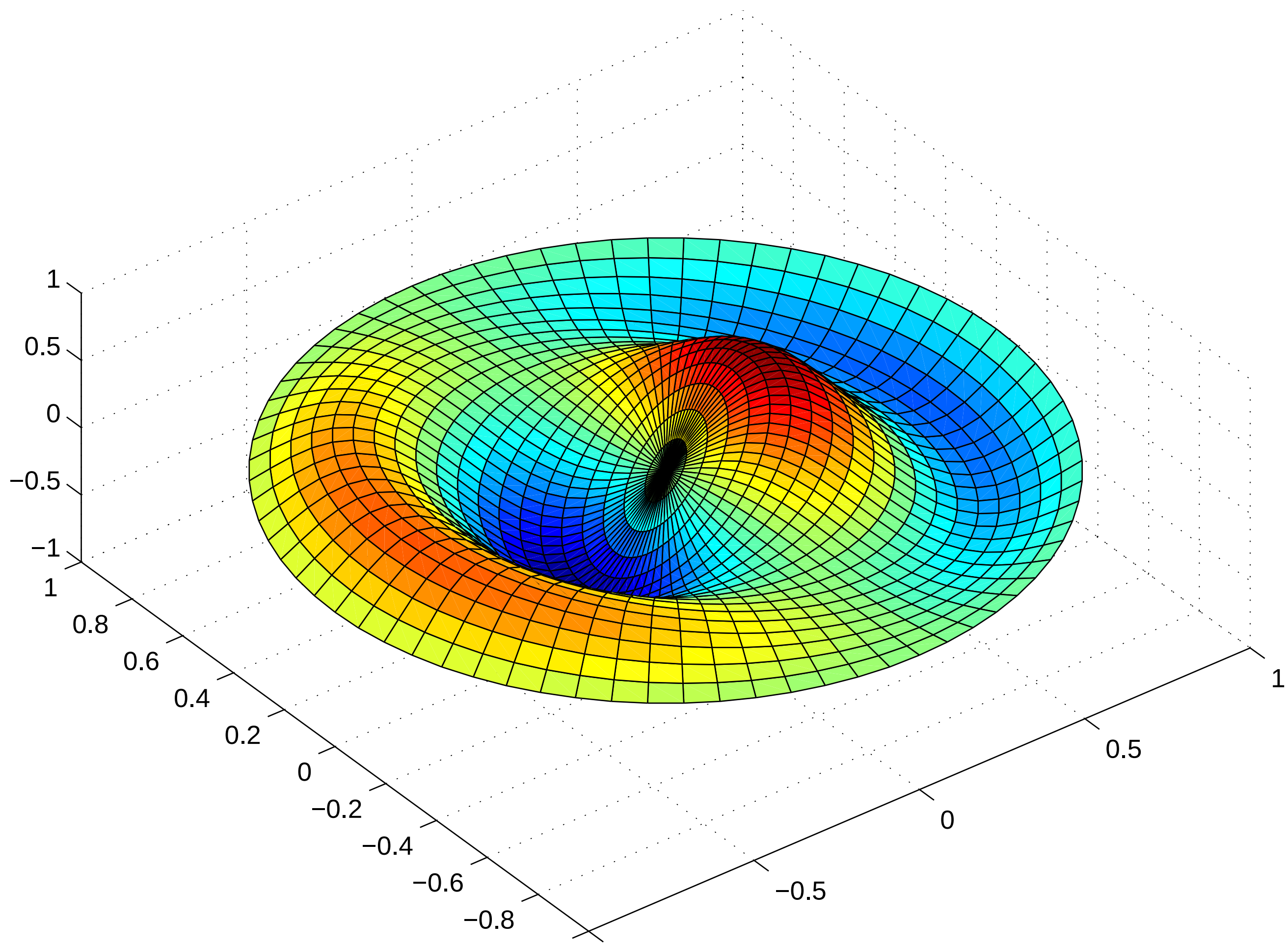
Fourth Eigen Mode
$J_0(5.520r)$ (second zero of J_0)



Fifth Eigen Mode
$J_3(6.380r)$ (first zero of J_3)



Sixth Eigen Mode
$J_1(7.016r)$ (second zero of J_1)



Seventh Eigen Mode
$J_4(7.589r)$ (first zero of J_4)