Lecture 14

TODAY

More on Damping Matrices and Modal Damping
Linear Viscoelasticity

Modal Damping Terms
RECALL

We saw the advantages of assuming modal damping, the question became: "How do we deduce that damping ζ_k associated with each mode?"

There are three standard answers.

1. **Experimentally. Plot frequency response, identify peaks with eigen frequencies, and determine fraction for critical damping from the half power points.**

2. **Where a real damping matrix is known, compute $P^T CP$ and discard the off-diagonal terms**

3. **Where the damping mechanism can be modeled, set the model damping to reproduce the dissipation ratio associated with modal vibration. This is the modal strain energy method.**

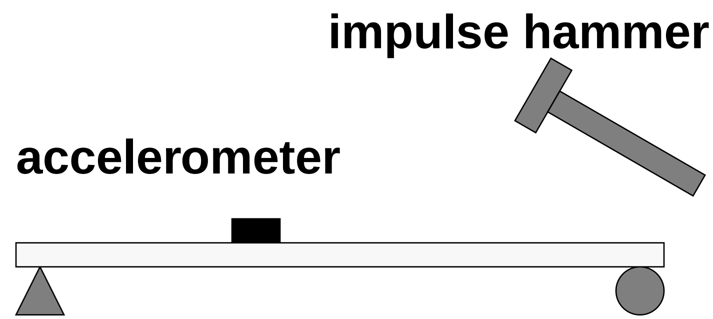
Examples and discussion follow.

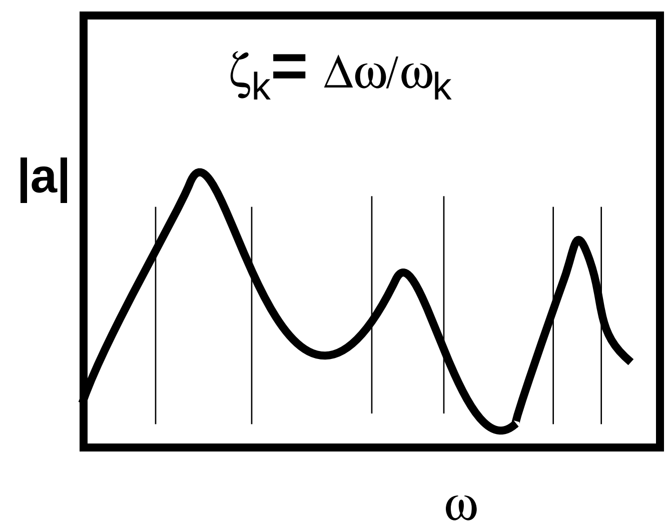
Damping from Experimental Data

Experimental measurements yield plots of acceleration versus frequency. Ideally, each peak is associated with a distinct natural mode of the system.

We say in the first portion of the test that we could deduce the damping of a single degree-of-freedom by examination of the “half power points”.

We can perform the same analysis for each of the modal peaks that are manifest from the experimental data, yielding a fractions for critical damping for those modes.





Modal Damping Terms from Full Damping Matrix

Say we are give a system

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In Matlab,

[P,D] = eig(k,m) :Find the generalized eigen solution:

PP =P/sqrt(P'*m*P) :Normalize the modal matrix

Cm = PP'*c*PP ;map the damping matrix to modal coordinates

```
 2.5000   -0.2887
-0.2887    0.8333
```

The diagonal of this matrix is $C^d = \begin{bmatrix} 2.500 & 0 \\ 0 & 0.833 \end{bmatrix}$

Modal Damping Terms from Full Damping Matrix

The full system in modal coordinates is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\beta}_1 \\ \ddot{\beta}_2 \end{bmatrix} + \begin{bmatrix} 2.500 & 0 \\ 0 & 0.833 \end{bmatrix} \begin{bmatrix} \dot{\beta}_1 \\ \dot{\beta}_2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 0.333 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Modal Damping Terms
By Modal Strain Energy Method

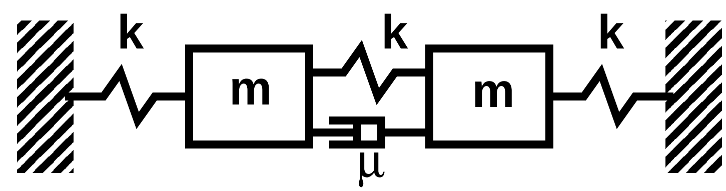
Given a system with mass matrix M and stiffness matrix K,

1. **compute the modal matrix P. Each eigen mode corresponds to a column of P.**
2. **compute the energy dissipation per cycle when the system is forced to undergo the deformation of the k'th eigen mode at the k'th eigen frequency**
3. **Calculate the maximum strain energy in that cycle.**
4. **Calculate the ratio of the dissipation to energy storage**
5. **Set the modal dissipation to reproduce the same ratio in modal coordinates.**

This requires and example

Modal Damping Terms
By Modal Strain Energy Method

Consider the system to the right, the mass and stiffness matrices are



$M = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **and** $K = k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$**. The algebraic eigen problem is**

$$\left(-\omega^2\frac{m}{k}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$ **, having eigen solutions**

$$\left(\left(\omega^2\left(\frac{m}{k}\right) = 1\right), \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right) \text{ and } \left(\left(\omega^2\left(\frac{m}{k}\right) = 3\right), \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\right)$$

Modal Damping Terms
By Modal Strain Energy Method

The energy dissipation associated with the dashpot is

$$D = \int_0^{2\pi/\omega} \mu(\dot{x}_2 - \dot{x}_1)^2 dt$$

For the first mode, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Re\left\{ Ae^{i\omega_1 t} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$,

$$\dot{x}_2 - \dot{x}_1 = Re\left\{ i\omega_1 Ae^{i\omega_1 t}(1/\sqrt{2} - 1/\sqrt{2}) \right\} = 0 \Rightarrow \zeta_1 = 0$$

The second mode will be more interesting.

Modal Damping Terms
By Modal Strain Energy Method

The energy dissipation associated with the dashpot is

$$D = \int_0^{2\pi/\omega} \mu(\dot{x}_2 - \dot{x}_1)^2 dt$$

For the second mode, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Re\left\{ Ae^{i\omega_2 t} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$,

$$\dot{x}_2 - \dot{x}_1 = Re\left\{ i\omega_2 Ae^{i\omega_2 t}(1/\sqrt{2} + 1/\sqrt{2}) \right\} = \sqrt{2}\omega_2 Re\{ iAe^{i\omega_2 t} \}$$

The dissipation is $D = 2\mu\omega_2\pi(A^*A)$.

Modal Damping Terms
By Modal Strain Energy Method

The energy storage is

$$V = \frac{k}{2}(Re\{Ae^{i\omega_2 t}\})^2 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \frac{3k}{2}(Re\{Ae^{i\omega_2 t}\})^2$$

and the max storage is $V_{max} = \frac{3k}{2}A^*A$**.**

The dissipation ratio is $\frac{D}{V_{max}} = \frac{2\mu\omega_2\pi(A^*A)}{\frac{3k}{2}A^*A} = \frac{4}{3}\frac{\mu\omega_2}{k}$

Modal Damping Terms
By Modal Strain Energy Method

The corresponding analysis in modal coordinates is

$$\ddot{\beta}_2 + 2\zeta\omega_2\dot{\beta}_2 + \omega_2^2\beta_2 = \mathcal{F}_2$$

Here, we assume that $\beta_2 = Re\{Be^{i\omega_2 t}\}$**.**

The energy dissipation per cycle is

$$D = \int_0^{2\pi/\omega_2} 2\zeta\omega_2(\dot{\beta}_2)^2 dt = \frac{\pi}{\omega_2}(2\zeta\omega_2)\omega_2^2(B^*B) = 2\pi\zeta\omega_2^2(B^*B)$$

The max energy stored per cycle is $V_{max} = \frac{\omega_2^2}{2}(B^*B)$ **so the dissipation ratio is** $4\pi\zeta$**.**

Modal Damping Terms
By Modal Strain Energy Method

We now set those damping ratios equal,

$$\frac{4}{3}\frac{\mu\omega_2}{k} = 4\pi\zeta_2 \Rightarrow \zeta_2 = \frac{\mu\omega_2}{3\pi k}$$

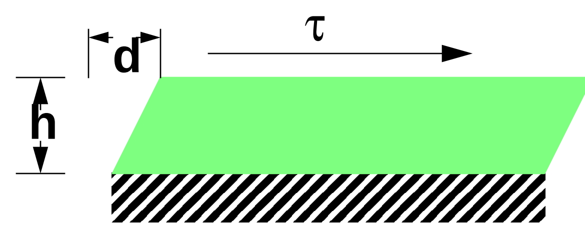
Our two modal equations are:

$\ddot{\beta}_1 + \omega_1^2\beta_1 = \mathcal{F}_1$ **and** $\ddot{\beta}_2 + 2\zeta_2\omega_2\dot{\beta}_2 + \omega_2^2\beta_2 = \mathcal{F}_2$

Introduction to Linear Viscoelasticity

Here we restrict our attention to rubbery polymers.

Such polymers have Poisson's ratio almost 0.5; we can ignore volumetric strains. In what follows, we restrict attention to problems for which all relevant deformations are in shear.



Engineering shear strain is defi as γ = d/h

The theory of linear viscoelasticity relates the current stress to the history of strain. The dependence is linear and time-translation invariant. This requires a constitutive equation of the sort:

$$\sigma(t) = \int_{-\infty}^{t} G(t-\tau)\dot{\gamma}(\tau)\,d\tau$$

where $G(\tau)$, the relaxation modulus, is a function of time characteristic of the material. There are sophisticated laboratory methods of measuring the relaxation modulus.

Introduction to Linear Viscoelasticity
Oscillatory Strain

For oscillatory deformation, the strain can be expressed

$\gamma(t) = A\sin(\omega t + \psi) = Im(\gamma_0 e^{i\omega t})$ **so** $\dot{\gamma}(t) = Im(i\omega\gamma_0 e^{i\omega t})$

We expect that the resulting stress will harmonic at the same frequency: $\sigma(t) = Im\{\sigma_0 e^{i\omega t}\}$**.**

These are substituted into the constitutive equation to find $\sigma_0 = \gamma_0(G'(\omega) + iG''(\omega))$ **where**

$$G'(\omega) = \int_0^\infty \omega G(\tau)\sin(\omega\tau)d\tau \,\&\, G''(\omega) = \int_0^\infty \omega G(\tau)\cos(\omega\tau)d\tau$$

Introduction to Linear Viscoelasticity
Oscillatory Strain

The stress associate with $\gamma(t) = A\sin(\omega t + \psi)$ is

$$AG'(\omega)\sin(\omega t + \psi) + AG''(\omega)\cos(\omega t + \psi)$$

Common laboratory techniques measure G' and G'' directly.

The storage modulus $G'(\omega)$ is an indicator of the ability of the polymer to store mechanical energy. The loss modulus $G''(\omega)$ is an indicator of the ability of the polymer to dissipate mechanical energy.

Introduction to Linear Viscoelasticity Superposition

Linear viscoelastic materials have the property of being linear.

The stress resulting from the superposition of strain histories is the superposition of the stresses associated with those histories.

If $\gamma(t) = \sum_k A_k \sin(\omega_k t + \psi_k)$**, then**

$$\tau(t) = \sum_k A_k G'(\omega_k) \sin(\omega_k t + \psi_k) + \sum_k A_k G''(\omega_k) \cos(\omega_k t + \psi_k)$$

Introduction to Linear Viscoelasticity
Energy Dissipation and Storage

At a given frequency ω, the energy dissipation per cycle per unit volume is

$$E_D = \int_0^{2\pi/\omega} \tau(t)\dot{\gamma}(t)\,dt$$

$$= A^2 \int_0^{2\pi/\omega} [G'(\omega)\sin\omega t + G''(\omega)\cos\omega t]\,\omega\cos(\omega t)\,dt$$

$$= \pi A^2 G''(\omega)$$

The max energy stored per unit volume in a cycle is estimated as

$$E_S = \frac{1}{2}\max[(A\sin\omega t)(AG'(\omega)\sin\omega t)] = \frac{1}{2}A^2 G'(\omega)$$

Introduction to Linear Viscoelasticity
Energy Dissipation and Storage

The loss factor η of either a structure or a material is defined by

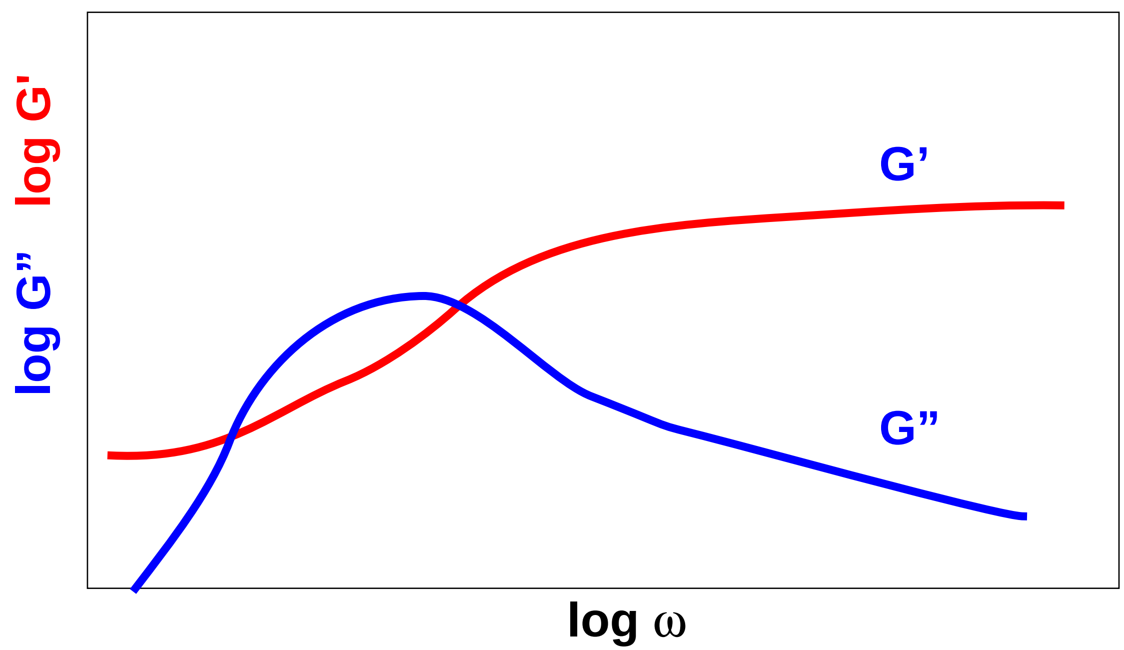
$$\eta(\omega) = \frac{E_D}{2\pi E_S}$$

. For a linearly viscoelastic material,

$$\eta(\omega) = G''(\omega)/G'(\omega) = \tan\delta(\omega)$$

The general forms of the storage modulus and loss modulus are illustrated in the figure

Loss modulus is maximum where storage modulus is steepest



Introduction to Linear Viscoelasticity
Solution of Transient Problems

Rigorous solution of transient problems is much more difficult. Using assumed modes, virtual work, and Lagrange Equations, we derive the following form of equation for structures of elastic and viscoelastic material

$$M\ddot{x} + Kx + \int_{-\infty}^{t} \Gamma(t-\tau)\dot{x}(\tau)d\tau = F(t)$$

where the matrix Γ is constructed of the relaxation moduli of the viscoelastic materials in the structure. This equation is very hard to solve in the time domain.

The form is simplified a little by resort to Laplace transforms: $s^2 MX(s) + (K + s\mathcal{L}(\Gamma))X(s) = L(F)$**,**

but inversion of this system of equations is usually impractical.

Viscoelasticity is Easier in the frequency domain

$$-\omega^2 M\bar{X}(\omega) + K\bar{X}(\omega) + \Gamma^*(\omega)\bar{X}(\omega) = \bar{F}(\omega)$$

Some finite element computer codes, including NASTRAN, can address this problem in frequency space. Again, inversion to the time domain can be a problem, but we often have an intrinsic interest in the frequency domain representation itself.

Correspondence Principle (Part One)

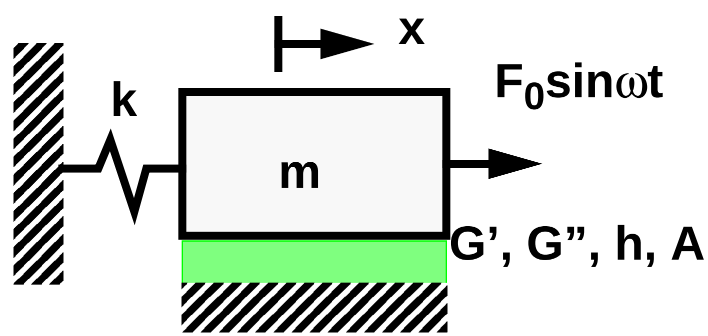
The governing equations in frequency space for a linearly viscoelastic structure are those of the corresponding elastic structure, but with elastic material properties replaced by complex material properties.

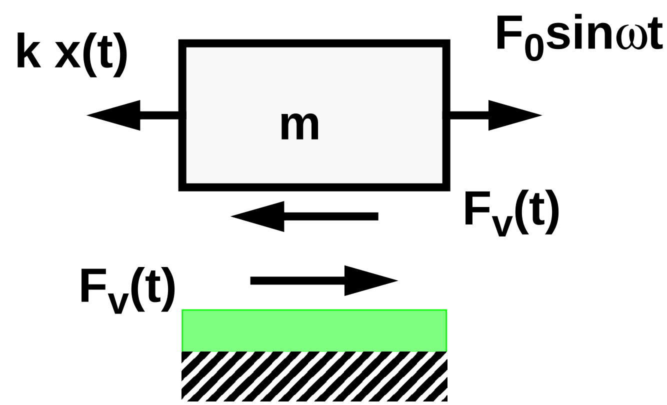
Correspondence Principle (Part Two)

This part is not as useful to us here. It observes that the solution to a quasi-static viscoelastic problem can be obtained by inversion of the Laplace transform mapped from the solution of a corresponding elastic problem.

Introduction to Linear Viscoelasticity
Steady State Oscillation

Steady State, harmonic response of a single degree of freedom system:





Consider the system shown. The mass sits on a pad of thickness h, and area A. At the imposed frequency, the storage and loss moduli are G'(ω) and G"(ω), respectively.

We anticipate that the steady state motion of the mass will be

$$x(t) = \bar{X}\sin(\omega t + \phi) .$$

In complex notation,

$F(t) = F_0 Im\{e^{i\omega t}\}$ **and**

$x(t) = \bar{X} Im\{e^{i(\omega t + \phi)}\} = Im\{X_0 e^{i\omega t}\}$ **where** $X_0 = \bar{X}e^{i\phi}$

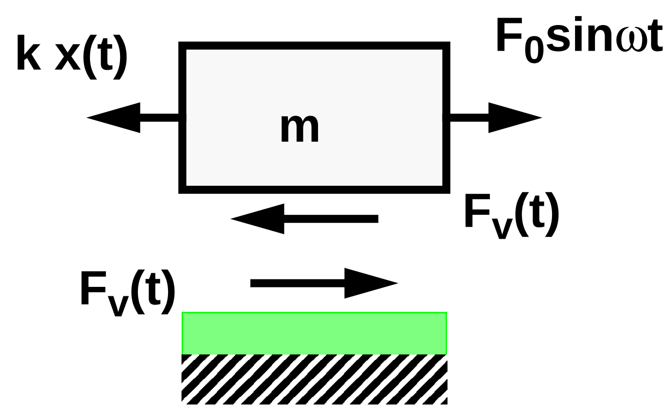
Introduction to Linear Viscoelasticity
Vibration of Linearly Viscoelastic Structures

The force applied to the polymer is

$$F_v(t) = \frac{\bar{X}A}{h}(G'(\omega)\sin(\omega t + \phi) + G''\cos(\omega t + \phi))$$
$$= \frac{A}{h}Im\{X_0 e^{i\omega t}(G'(\omega) + iG''(\omega))\}$$

The equation of motion of that mass is

$m\ddot{x} = F_0 \sin\omega t - kx(t) - F_v(t)$.



Making all the appropriate substitutions, we find

$$-\omega^2 m X_0 = F_0 - kX_0 - X_0\frac{A}{h}(G' + iG'')$$

Note the similarity to the elasto-dynamics problem.

$$-\omega^2 m X_0 = F_0 - kX_0 - X_0 \frac{A}{h}(G' + iG'')$$

From which we can solve for the complex frequency response function for normalized displacement as a function of frequency:

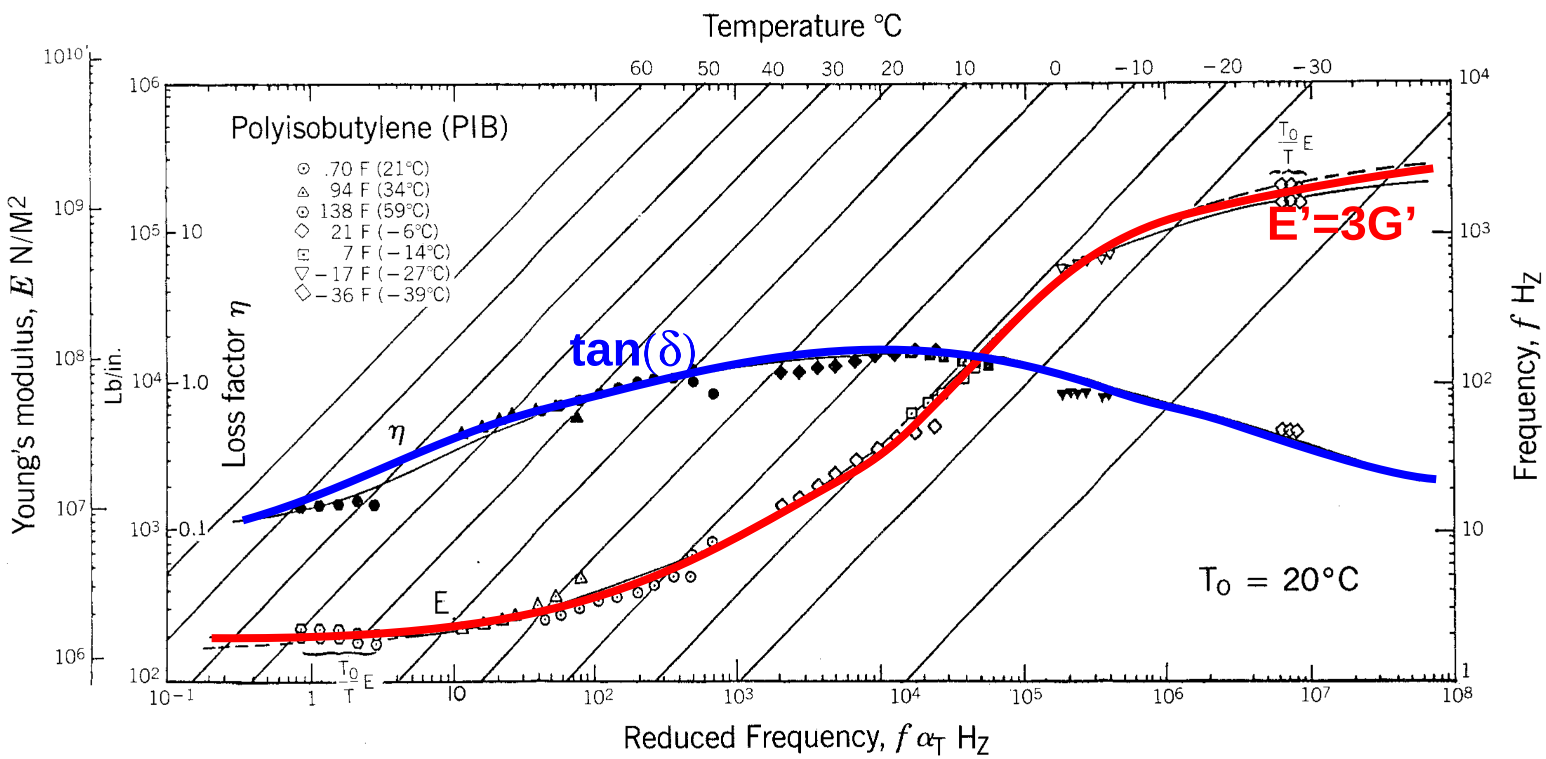
$$\frac{X_0}{F_0}(\omega) = \frac{1}{[k + (A/h)G'(\omega) - m\omega^2] + i(A/h)G''(\omega)}$$

Note that $\frac{X_0}{F_0}(\omega \to \infty) = \frac{1}{-\omega^2 m}$**, as it should. This is mass loading.**

Also note that $\frac{X_0}{F_0}(0) = \frac{1}{k + (A/h)G'(0)}$**, as it should. This is the static response.**

Introduction to Linear Viscoelasticity
A Material

Lets examine the response of a real system. The following plot for polyisobutylene is taken from "Vibration Damping" by Nashif et. al.



011C. Nomogram.

We can fit the above plots reasonably well in $0.1 < \frac{\omega}{2\pi} < 10^5$ **by**

$$Log(3G'(\omega)) = Log(E'(\omega)) = 6.2 + 0.0112\left(Log\left(\frac{\omega}{2\pi}\right)\right)^{3.3}$$ **or**

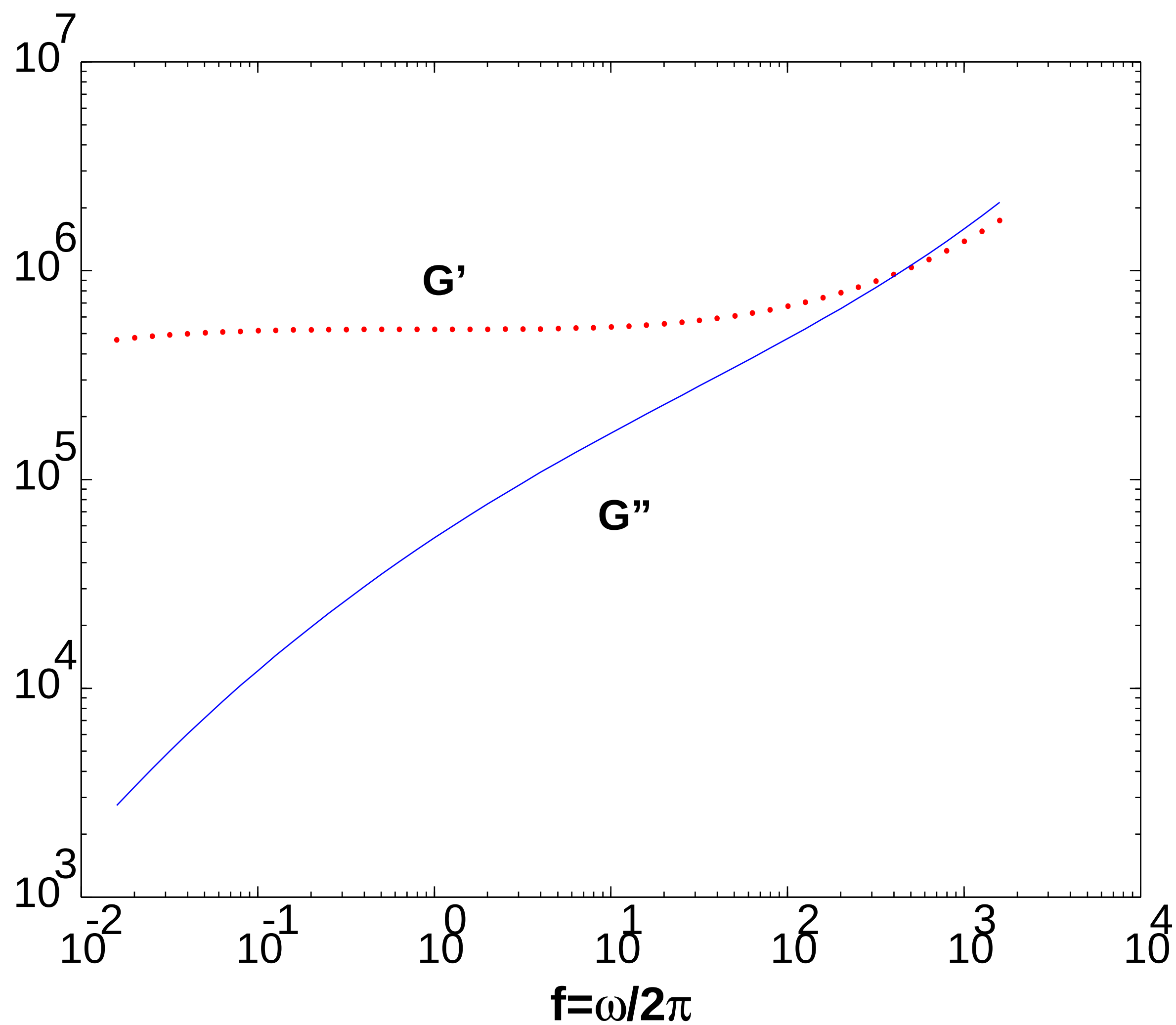
$$G'(\omega) = 10^{5.72 + 0.0112\left(Log\left(\frac{\omega}{2\pi}\right)\right)^{3.3}}$$ **and**

$$Log(\tan\delta(\omega)) = -1 + 0.56Log\left(\frac{\omega}{2\pi}\right) - 0.069\left(Log\left(\frac{\omega}{2\pi}\right)\right)^2$$ **so**

$$G''(\omega) = G'(\omega)\tan\delta(\omega)$$

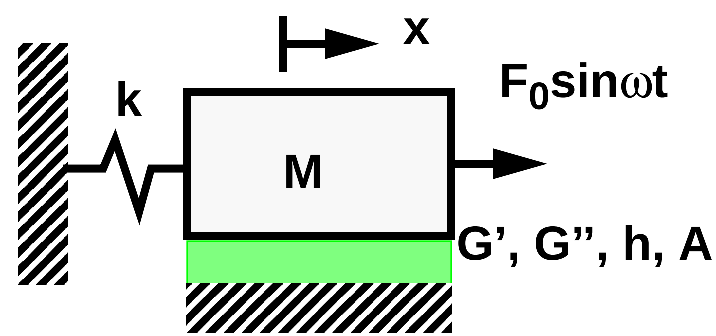
$$= 10^{4.72 + 0.56Log\left(\frac{\omega}{2\pi}\right) - 0.069\left(Log\left(\frac{\omega}{2\pi}\right)\right)^2 + 0.0112\left(Log\left(\frac{\omega}{2\pi}\right)\right)^{3.3}}$$

The above approximations show the correct behavior at low frequencies



Lets assign properties to the problem we discussed above.

Say our mass is 0.10m x 0.10m in area, 0.1m high, and made of steel.



$$A = (0.1m)(0.1m) = 0.01m^2$$

$$M = (0.1m)(0.1m)(0.1m)(7700kg/m^3) = 7.7kg$$

Set $h = 0.01m$ and $k = 3E6N/m$

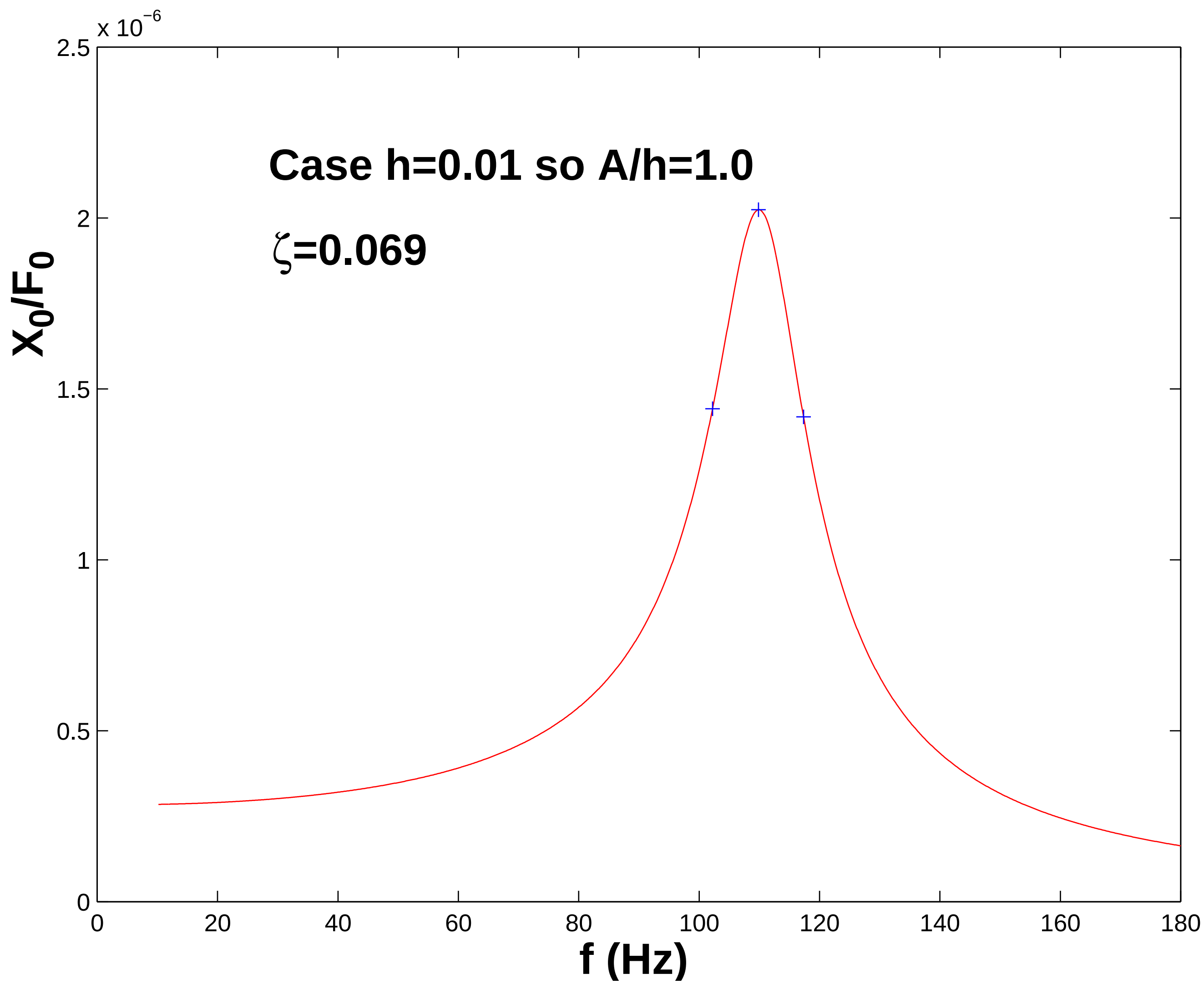
Then $A/h = 1.0$

We shall calculate the fraction of critical damping by two methods:

- Examination of half-power band width. This reflects the true material properties.
- Modal strain energy. This is approximate.

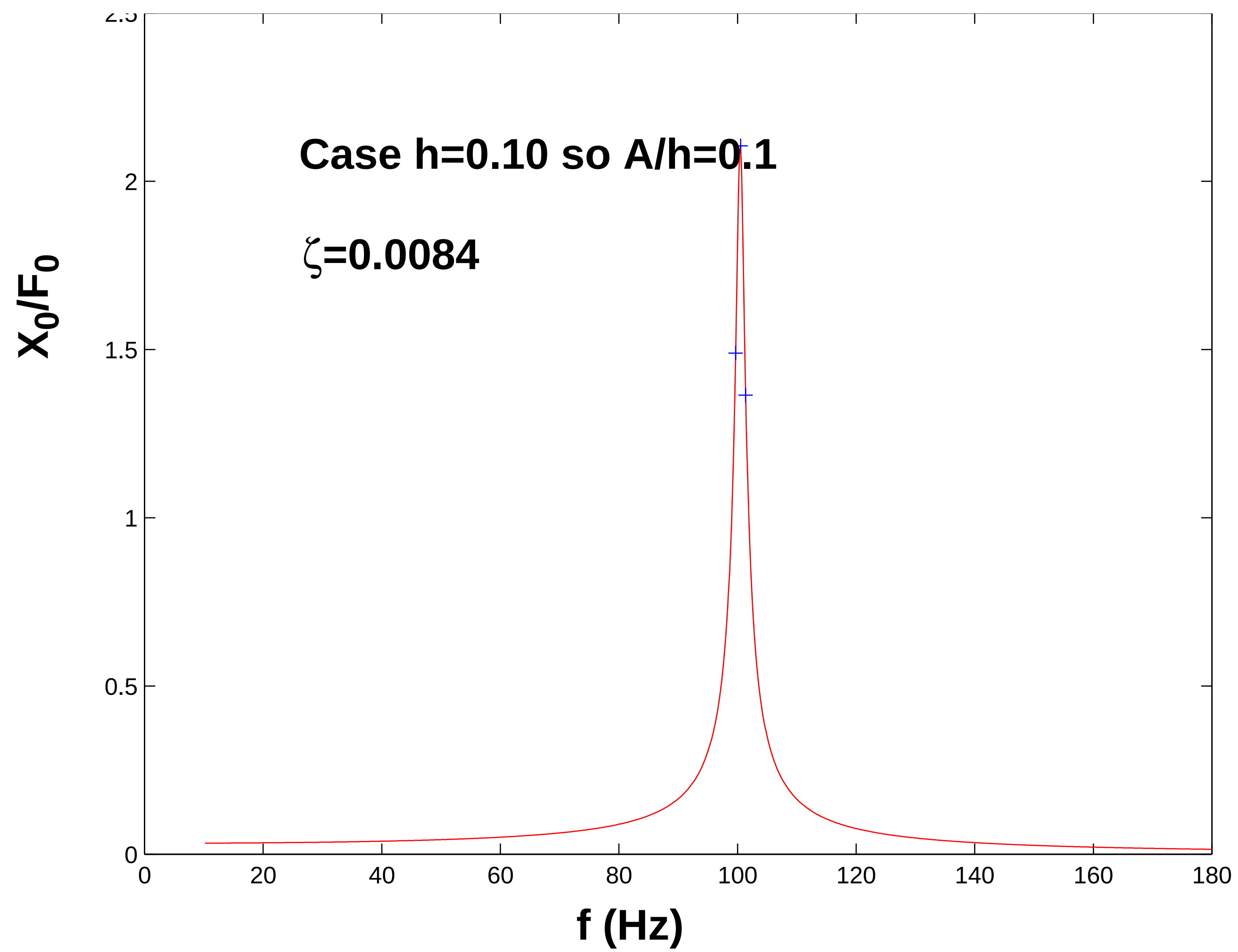
Fraction of Critical Damping in Viscoelastic Structure

Lets calculate the damping by examination of the half power points.



Fraction of Critical Damping in Viscoelastic Structure

Lets consider another case:



Introduction to Linear Viscoelasticity

The above treatment is that for a steady state oscillatory response.

To solve this system in the time domain, we must

- **solve an integro-differential equation (usually not a good idea)**
- **approximate this problem by a second order system with constant coefficients. This is what modal strain energy was made for.**

Introduction to Linear Viscoelasticity
Solution of Transient Problems

In order to solve transient problems involving linearly viscoelastic materials, we need a simpler approach. The approach that has developed over the last four decades is:

- **Map the elastic problem to mode space**
- **Estimate modal damping via the modal strain energy method**
- **Integrate the resulting modal equations**
- **Map displacements back to spacial degrees of freedom**

We shall see some examples using modal strain energy to estimate modal damping for viscoelastic structures.

Introduction to Linear Viscoelasticity
Damping from Modal Strain Energy

Lets see what damping model we get from the modal strain energy method applied to our earlier problem.

For the elastic system, $\omega_n = (k/m)^{1/2} = 624$ **/sec =>**

$f_n = 100$ **Hz. Lets look at the energy dissipation per cycle of this single degree of freedom system when oscillating at its resonant frequency at amplitude B:**

$$E_D = \pi B^2 (A/h) G''(\omega_n) = \pi B^2 (4.74 \times 10^5 N/m)$$

The energy stored per cycle is

$$E_S = \frac{1}{2} B^2 (k + G'(\omega)(A/h)) = \frac{1}{2} B^2 \, 3.07 \times 10^6 N/m$$

$$2\zeta = \eta = \frac{E_D}{2\pi E_S} = 0.134 \Rightarrow \zeta = 0.067$$

Introduction to Linear Viscoelasticity
Approximation with 2nd Order System

Our approximate model for the viscoelastic system is

$$\ddot{\beta} + 2\zeta\omega_n\dot{\beta} + \omega_n^2\beta = F/\sqrt{M}$$

We convert back to spacial coordinates:

$$M\ddot{x} + 2\zeta\sqrt{KM}\dot{x} + Kx = F(t)$$

In order to recover the static response, the stiffness we use is that computed at frequency $\omega = 0$:

$$K = k + (A/h)G'(0) = 3.54\times10^6 N/m.$$

We just found the found the fraction of fraction of critical damping $\zeta = 0.067$, and the mass remains $M = 7.7kg$

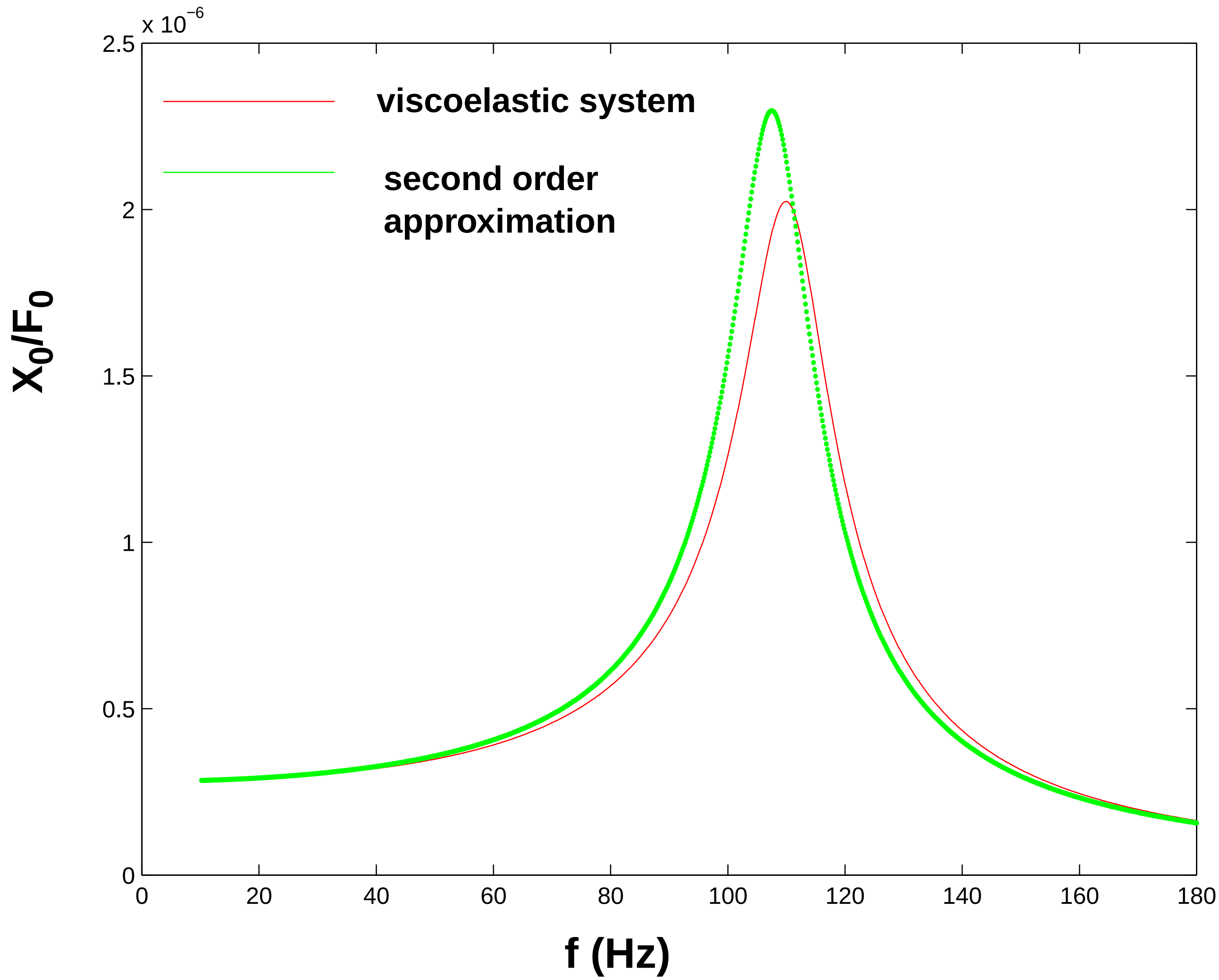
introduction to Linear Viscoelasticity
Approximation with 2nd Order System

Lets compare this approximation to the actual system for the case of steady state harmonic excitation: $F = Im\{F_0 e^{i\omega t}\}$ **and** $x = Im\{X_0 e^{i\omega t}\}$ **yielding**

$$X_0 / F_0 = \frac{1}{[(K - M\omega^2) + i2\zeta\omega\sqrt{KM}]}$$

Lets compare this approximation to the full viscoelastic system.

Comparison of Linear Viscoelastic Model with Second Order System



Matlab code for the above:

```
% Calculate response of systems in slides page 28 and page 35
% of Lecture 14
% spring stiffness and the mass
k=3e6
m=7.7
%
% Repeat current value of Ah
Ah
%
% specify the frequency range to examine
fmin = 10.0; fmax = 180.0;
Points = 1001;
 f(i) =fmin + (fmax-fmin)*(i/Points);
 w(i) = f(i)*2*pi;
 %
 % Interpolating formulae for storage and loss moduli
 gp(i) = real(10.0^(5.72 + 0.0112*(log10(f(i)))^3.3));
 gpp(i) = real(10.0^(4.72+0.56*log10(f(i)) -0.069*(log10(f(i)))^2 +
0.0112*(log10(f(i)))^3.3));
 % Driving point frequency response.  A complex function of
frequency
 H(i) = 1.0/(k+Ah*gp(i)-m*w(i)^2 + j*Ah*gpp(i));
end
%
Ha = abs(H); % magnitude of the driving point frequency
response
%
% Power is proportional to the square of displacement
amplitude.
% We shall find the half power points
ic = [1 Points]; % array of indices of half power points
i1 = 1; i2=Points;
 Hmax = max(Ha);
 % find the frequency where Ha is maximum
 i0 = find( abs(Ha - Hmax)<1.0e-10);
 % find the half power points
 cross=0;
 for i=2:Points
 if( (Ha(i)-Hmax/2^0.5)*(Ha(i-1)-Hmax/2^0.5) <0 )
  cross = cross+1;
  ic(cross) = i;
 end
end
%
%mark these points
f0 = f(i0); f1  =  f(ic(1)); f2 =  f(ic(2));
h0 = Ha(i0);h1  = Ha(ic(1)); h2 = Ha(ic(2));
%
%%evaluate fraction of critical damping
zeta =  (f2 - f1)/(2*f0)
%
%%try modal strain energy method
Ed = pi*Ah*gpp(i0);
Es = 0.5*(k+gp(i0)*Ah);
eta = Ed/(2*pi*Es);
zetaE = eta/2
%
% Plot the magnitude of driving point frequency response and
%   mark the points
p1 =  plot(f,Ha,'r-', f0,h0,'b+', f1,h1,'b+', f2,h2,'b+');
%
%
%plot the modal approximation to the frequency response
for i=1:Points
 HM(i) = 1.0/(k+Ah*gp(1)-m*w(i)^2 + j*(2*zetaE*m*w(i)*(k/
m)^0.5));
 HMa(i) = abs(HM(i));
end
% Plot it and compare with the ViscoElastic model
p2 = plot(f,Ha,'r-', f, HMa, 'g.');
```

Vibration of Linearly Viscoelastic Structures
Summary of Modal Strain Energy Method

1. **Perform eigen solution of linear elastic system, calculating modes and frequencies.**
2. **For each mode, calculate the elastic energy E_S stored in the system when it is deformed to the extreme of each mode. For viscoelastic parts of the structure, use the storage modulus at those eigen frequencies.**
3. **For each mode, calculate the energy E_D dissipated per cycle.**
4. **For each mode, set** $\zeta_k = \dfrac{E_D}{2\pi E_S}$

Next Time

Introduction to Frequency Response Methods