Slides of Lecture 2

Today's Class

- **Homework**
- **Assembly of Systems from Components**
- **Convolutions with Indicial and Impulse Response Functions**
- **Beginning of Fourier Analysis and the Complex Operators**

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Recalling Homework Number 1

1. Solve the transient Single DOF problem

$y''(\tau) + 2\zeta y'(\tau) + y(\tau) = \sin(\tau\alpha)$ **numerically using MATLAB until steady state for values of** α **equal to 0 to 2.0 in increments of 0.1.**

2. Plot $\tilde{H}(\alpha)$ **versus** α

3. plot $\tilde{\psi}(\alpha)$ **versus** α

4. Do this for the three values of ζ**: 0.01, 0.1, 1.0**

Hint: Let $v = \begin{bmatrix} y' \\ y \end{bmatrix}$ **and solve** $v' + \begin{bmatrix} 2\zeta & 1 \\ -1 & 0 \end{bmatrix} v = \begin{bmatrix} \sin(\alpha\tau) \\ 0 \end{bmatrix}$

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Solution
Having finished reading Chapter 1 of text

The transfer function $\tilde{H}$ is $\tilde{H} = \dfrac{1}{1 - \alpha^2 - i2\zeta\alpha}$

The magnitude of $\tilde{H}$ is determined from

$$|\tilde{H}|^2 = \tilde{H}\tilde{H}^* = \frac{1}{(1-\alpha^2)^2 + (2\zeta\alpha)^2}$$

and the phase is determined from

$$\tilde{\psi}(\alpha) = \operatorname{atan}\frac{Im(\tilde{H})}{Re(\tilde{H})} = \operatorname{atan}\frac{(\tilde{H} - \tilde{H}^*)}{i(\tilde{H} + \tilde{H}^*)} = \operatorname{atan}\frac{2\alpha\zeta}{1-\alpha^2}$$

These are plotted as follows:

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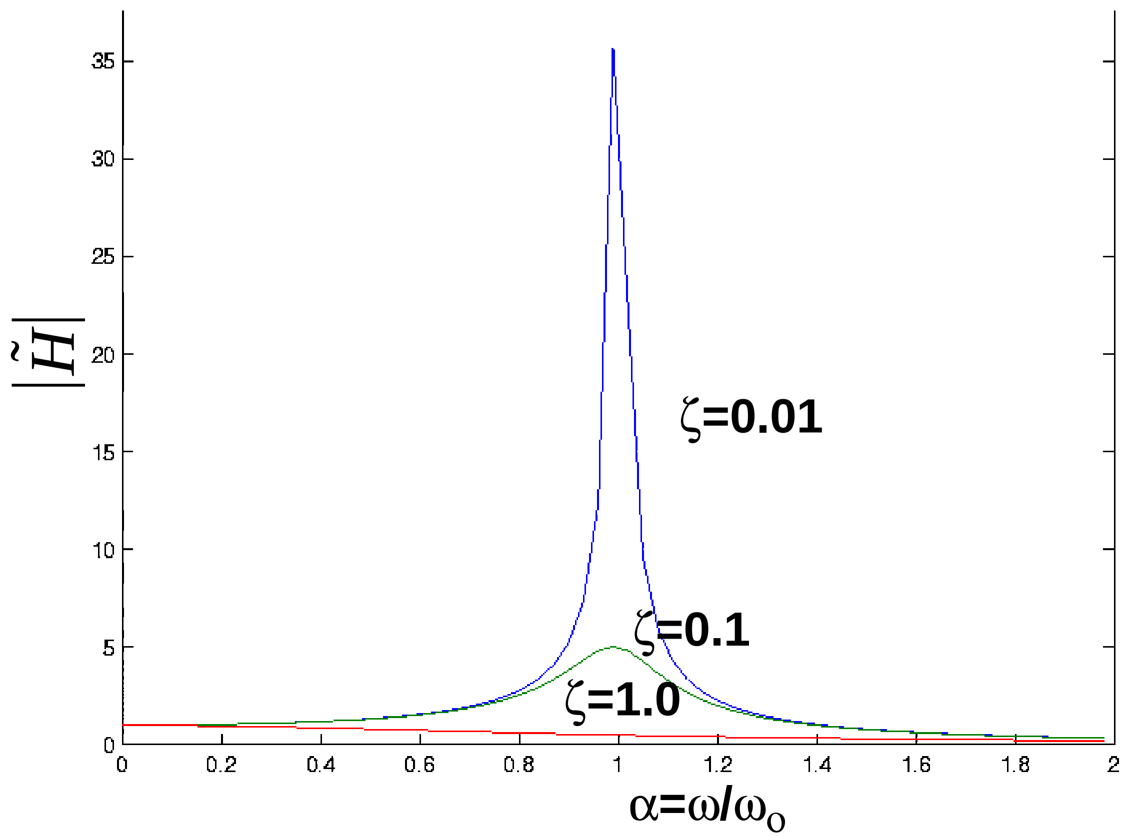
Plots from Homework 1
Magnitude of Transfer Function



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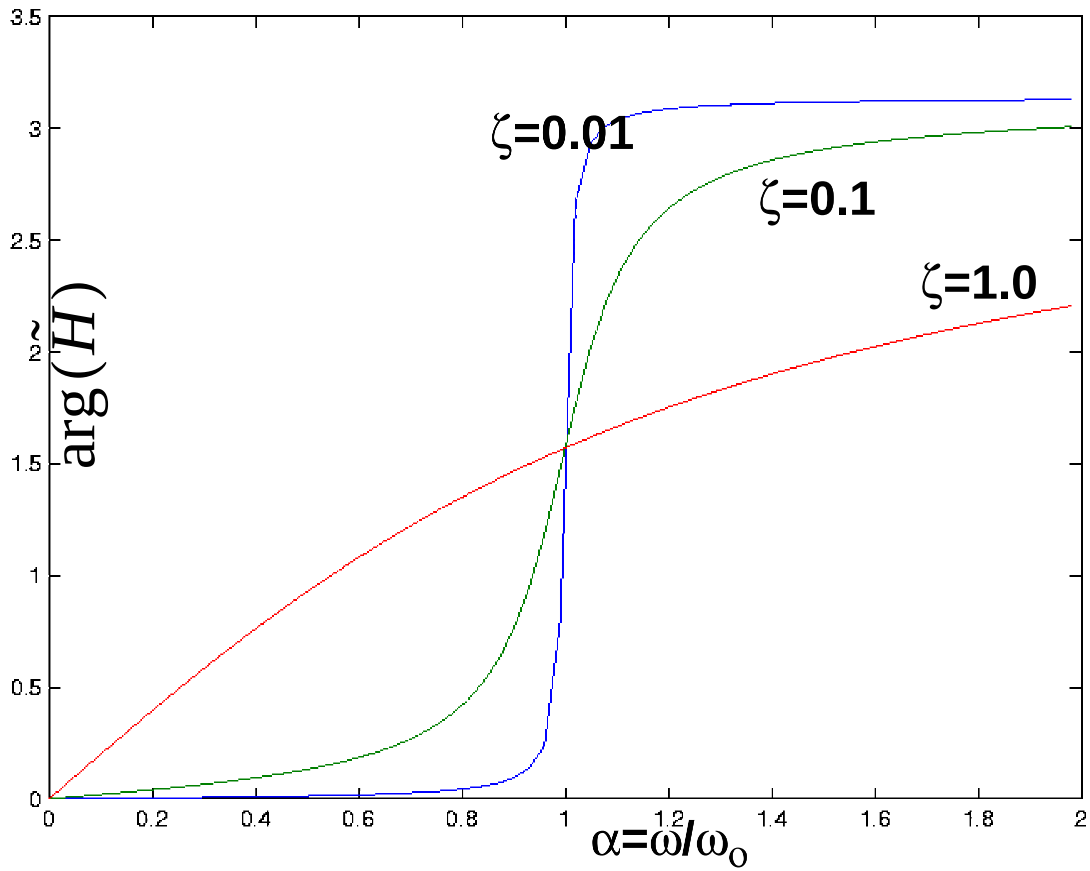
Plots from Homework 1
Phase of Transfer Function



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Conclusions from Homework 1

- **There are advantages to frequency analysis. We shall see more of this later.**
- **It takes a slightly damped system a very long time to reach steady state.**

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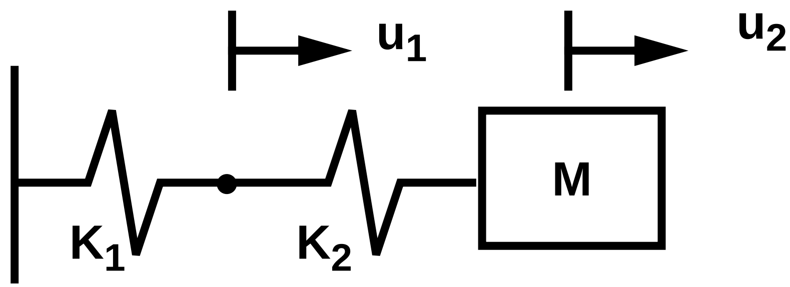
Synthesis of Systems

Recall that it was asserted that:

Intrinsic # DOF of System = #Masses X DOF/Mass

Consider the following system



Does this system have two degrees of freedom?

Should it have just one?

How do we reconcile this?

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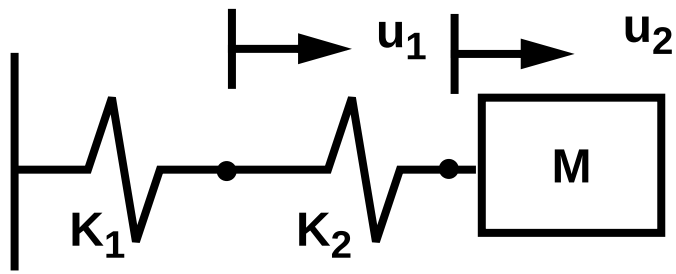
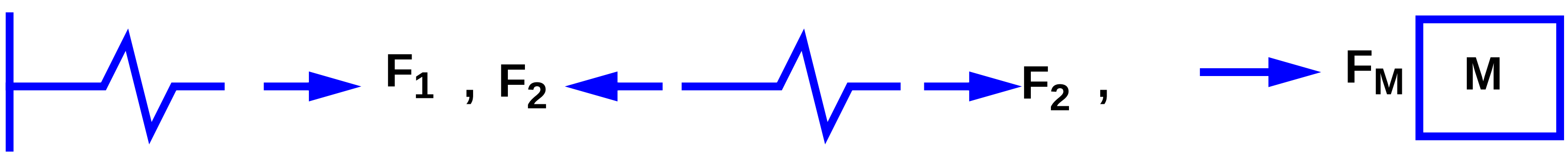
Synthesis of Systems

Resolution is achieved through the elimination of massless degrees of freedom - statically.





As shown here, F_1 and F_2 are *applied* forces: $F_1 = K_1 u_1$,

$F_2 = K_2(u_2 - u_1)$. **F_M is the force *applied* to the mass** $F_M = M\ddot{u}_2$

Balancing forces at a virtual mass at location x1 to deduce $F_1 = F_2$.

Balancing forces at the actual mass at location x2, $F_M = -F_2$

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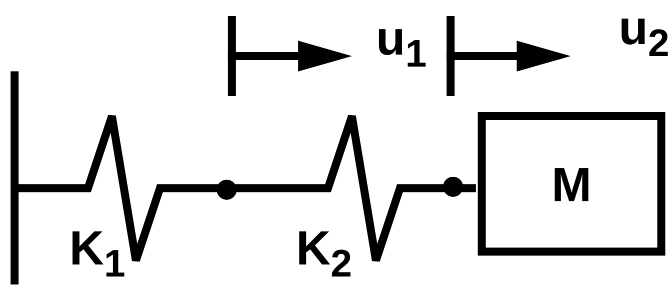
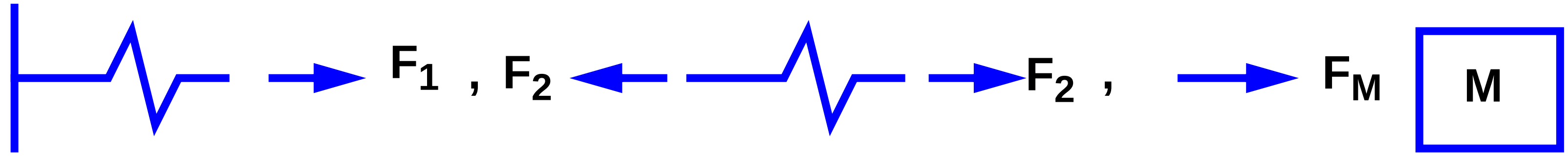
Synthesis of Systems





From $F_1 = F_2, u_1 = \dfrac{K_2}{K_1 + K_2} u_2$ **and** $F_1 = F_2 = \dfrac{K_1 K_2}{K_1 + K_2} u_2$

From $F_M = -F_2, M\ddot{u}_2 = -\dfrac{K_1 K_2}{K_1 + K_2} u_2$ **which is our desired single degree-of-freedom system.**

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Synthesis of Systems
Stating the Obvious

- **The reaction forces on massless subsystems is equal and opposite to the forces applied to them.**
- **Note that extra degrees of freedom are removed by balancing forces and equating displacements.**
- **Constitutive expressions relate applied forces to resulting kinematics.**

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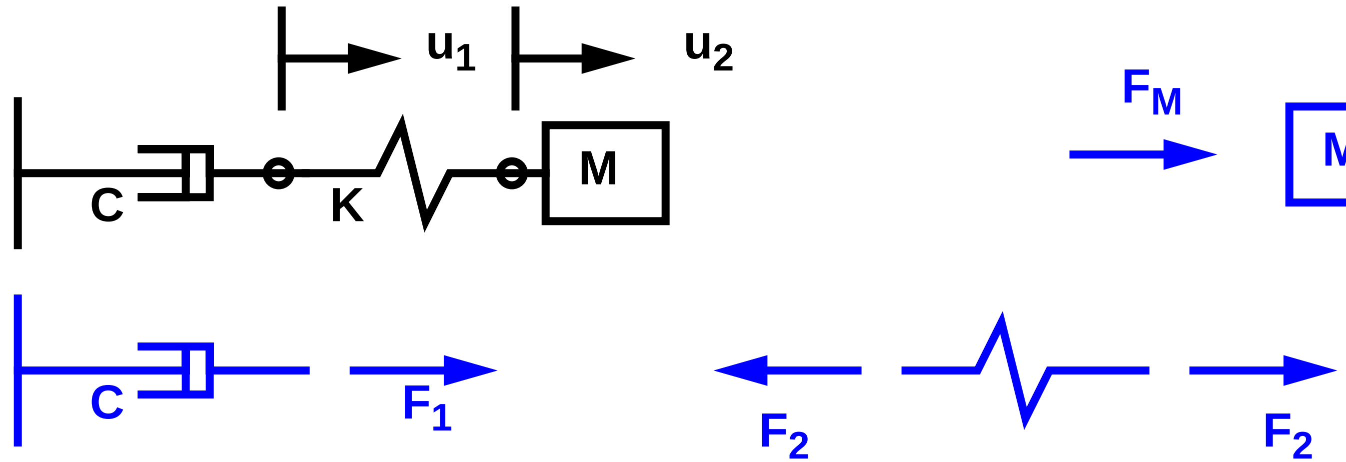
Synthesis of Systems
Some More Complex Cases

Returning to the liquid-like second order system



Recall the synthesis rules:

- **The displacements at location 1 are implicitly continuous.**
- **The forces balance: $F_1 = F_2$, and $F_2 = -F_M$. Remember that the constitutive equations relate *applied* force to resulting kinematics.**

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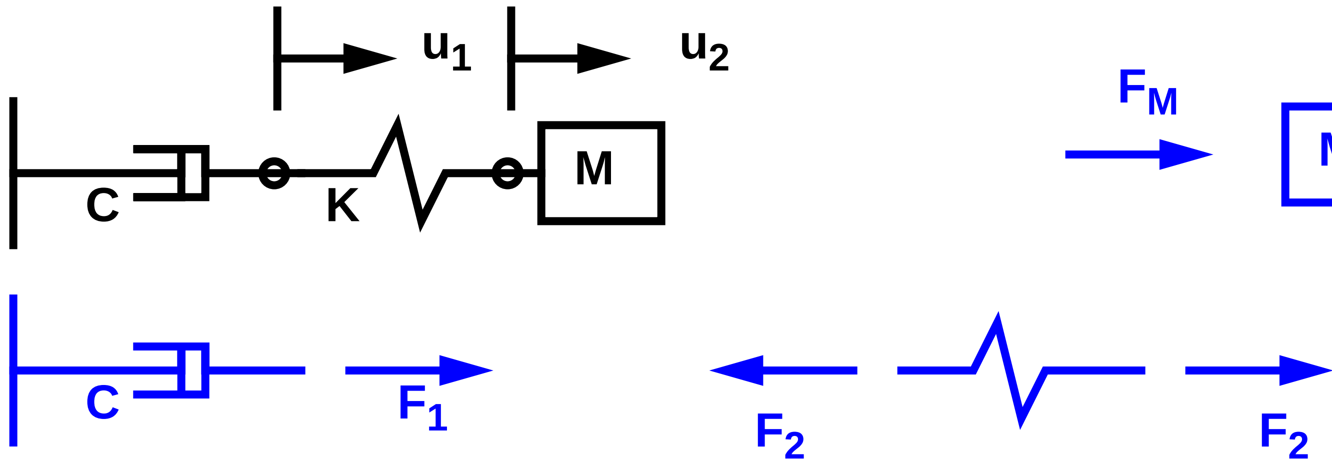
Synthesis of Systems
Some More Complex Cases

Returning to the liquid-like second order system



Recall the constitutive rules:

- $F_1 = c\dot{u}_1$
- $F_2 = K(u_2 - u_1)$
- $F_M = M\ddot{u}_2$

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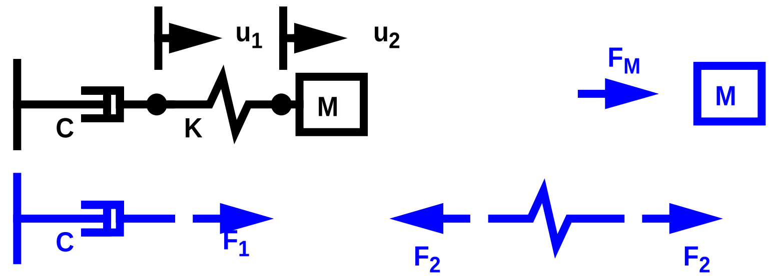
Synthesis of Systems
Some More Complex Cases

Solving the system:



$c\dot{u}_1 = K(u_2 - u_1)$ **and**

$K(u_2 - u_1) = -M\ddot{u}_2$.

Setting $\lambda = c/K$, **we solve the**

first to find: $u_1(t) = \int_0^t e^{-(t-\tau)/\lambda} u_2(\tau)\frac{d\tau}{\lambda}$

which is substituted in to the dynamics equation to obtain the

governing equation $M\ddot{u}_2 + K\left(u_2 - \int_0^t e^{-(t-\tau)/\lambda} u_2(\tau)\frac{d\tau}{\lambda}\right) = 0$.

This is tedious.

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Synthesis of Systems Using Laplace Transforms

$c\dot{u}_1 = K(u_2 - u_1)$ **and** $K(u_2 - u_1) = -M\ddot{u}_2$

Becomes: $csU_1(s) - cu_1(0) = K[U_2(s) - U_1(s)]$ **and**

$$K[U_2(s) - U_1(s)] + s^2 MU_2(s) = s\dot{u}_2(0) + u_2(0)$$

After solving out $U_1(s)$**, we have**

$$\left[s^2 M + K\frac{cs}{K + cs}\right]U_2(s) = K\frac{cu_1(0)}{K + cs} + M[s\dot{u}_2(0) + u_2(0)].$$

It is good practice to normalize:

$$\left[s^2 + \omega_n^2\frac{\lambda s}{1 + \lambda s}\right]U_2(s) = \omega_n^2\frac{\lambda s}{1 + \lambda s} + [s\dot{u}_2(0) + u_2(0)]$$

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Synthesis of Systems
Another Problem

Remember that we were told that

Intrinsic # DOF of System = #Masses X DOF/Mass

That was a lie.

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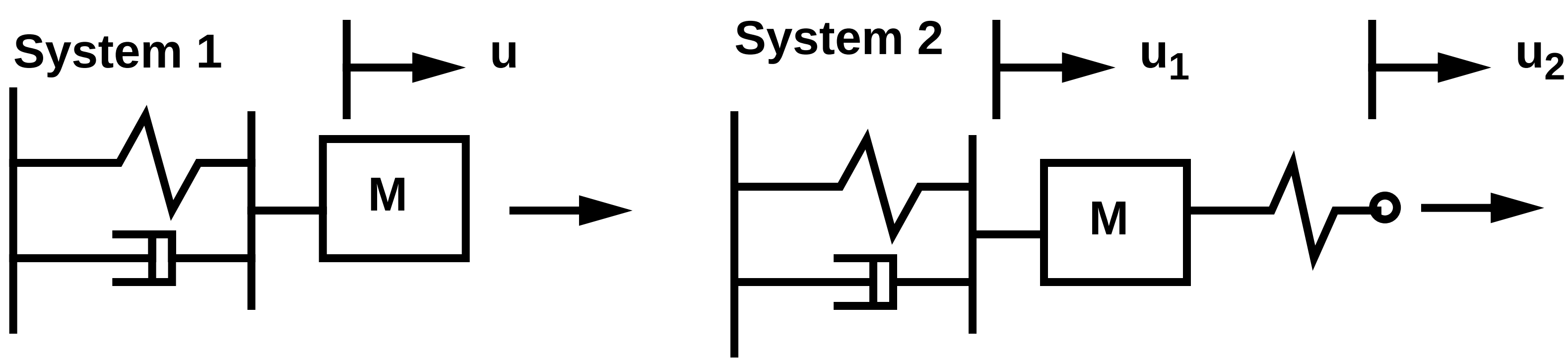
Synthesis of Systems
Another Problem

So far today, we have considered homogeneous systems - those without imposed forces.

Lets consider the following two systems:



Where boundary forces or displacements are applied at massless degrees of freedom, we often retain those degrees of freedom.

Often we resolve out degrees of freedom associated with the masses.

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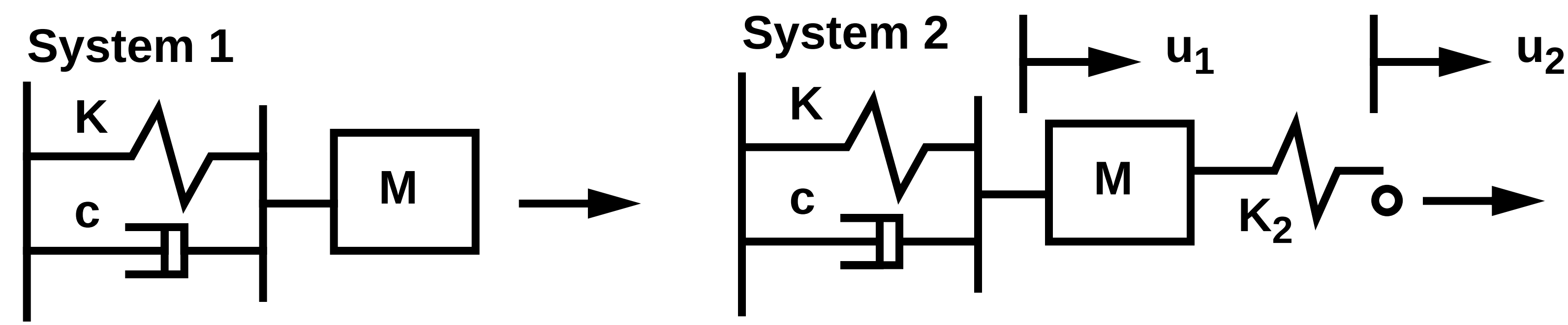
Synthesis of Systems
Another Problem



Governing equations for System 1: $M\ddot{u} + c\dot{u} + Ku \;=\; F$

For System 2: $M\ddot{u}_1 + c\dot{u}_1 + Ku_1 \;=\; K_2(u_2 - u_1)$

$$F \;=\; K_2(u_2 - u_1)$$

Lets solve these subject to homogeneous initial conditions

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Synthesis of Systems
Impulse Response

Solve System 1: $M\ddot{u} + c\dot{u} + Ku = F$**:**

Using Laplace transforms, $U(s) = \mathcal{L}(F)(s)/(Ms^2 + cs + K)$

Transforming back to the time domain,

$$u(t) = \int_0^t h(t-\tau)F(\tau)d\tau \text{ .}$$

where $h(t) = \frac{1}{M}\mathcal{L}^{-1}\left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$

h **is the impulse response function for this system.**

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Synthesis of Systems
Impulse Response Function

If we have a system whose governing equation in response to a imposed force is $L(u)(t) = F(t)$**, having a Laplace transform** $G(s)\mathcal{L}(u)(s) = \mathcal{L}(F)(s)$**, then**

$$h = \mathcal{L}^{-1}(G)$$

is the impulse response function for that system. The resulting displacement is $u(t) = \int_0^t h(t-\tau)F(\tau)d\tau$

Note: h **solves** $L(h)(t) = \delta(t)$

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Synthesis of Systems
The Integral of Duhamel

Similarly, if we define the indicial response function g:

$$u(t) = \int_0^t g(t-\tau)\dot{F}(\tau)d\tau + F(0)g(t)$$

$g(t)$ **solves the equation** $L(g)(t) = H(t)$**, where** H **is the Heavyside step function.**

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Homework Number 2

Part A. Determine the impulse response function $h(t)$ for System 1. Assume that the system is under-damped ($\zeta < 1$).

Part B. Determine the indicial response function $g(t)$ for System 1.

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Homework Number 3

Part A. Resolve out the degree of freedom u1 associated with the mass in System B to in favor of u2. Obtain a governing equation in a single degree of freedom u2. You may leave this in Laplace transform space.

Part B. Determining the impulse response function for this problem. This function lives in the time domain. Assume that the system is under-damped.

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Concepts from Complex Variable Analysis
A Reminder

General Notions

Complex number $z = x + iy$

Real and Imaginary operators: $Re\{x + iy\} = x$, $Im\{x + iy\} = y$

Complex Conjugate: $(x + iy)^* = x - iy$

Observe: $Re\{z\} = \frac{1}{2}(z + z^*)$ **and** $Im\{z\} = \frac{1}{2i}(z - z^*)$

Complex Function of a Complex Variable: $f(z) = u(z) + iv(z)$

An Analytic Function has a Unique Derivative:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial (iy)} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

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Concepts from Complex Variable Analysis
A Reminder

About Analytic Functions

$z = x + iy$ **is analytic**

z^n **is analytic if n is an integer**

$e^{\alpha z}$ **is analytic**

if f_1 **and** f_2 **are each analytic, then so are**

$$\alpha f_1, f_1 + f_2, \ f_1 f_2 , f_1 / f_2$$

if f **is analytic, then so is** f'

if $Re(f)$ **is constant then** $Im(f)$ **is also constant.**

if $Im(f)$ **is constant then** $Re(f)$ **is also constant.**

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