Lecture 14.5

COMPLEX MODES

State Space Formulation

What happens when we cannot assume modal damping?

As usual, we assume a second-order system with constant coefficients:

$$M\ddot{x} + C\dot{x} + Kx = F(t),$$

Put this into a state space formulation:

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \dot{\begin{bmatrix} \dot{x} \\ x \end{bmatrix}} + \begin{bmatrix} C & K \\ -M & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} F(t) \\ 0 \end{bmatrix}$$

Each of the if the original problem has N degrees of freedom, the new system has $2N$ degrees of freedom.

Change of Variables

Define $\mathcal{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$. Since M is nonsingular, so is $\mathcal{M}$.

Define $y = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}$, substitute the change of varibles into the state-space governing equation, and premultiply by $\mathcal{M}^{-1}$ toobtain

$$\dot{y} - Ay = \tilde{F} \text{ where } A = -\mathcal{M}^{-1}\begin{bmatrix} C & K \\ -M & 0 \end{bmatrix} \text{ and } \tilde{F} = \mathcal{M}^{-1}\begin{bmatrix} F(t) \\ 0 \end{bmatrix}$$

We shall use modal analysis and orthogonality to solve the above first order system.

Bi-Orthogonality

Because, the matrix A is non-symmetric its eigen vectors are not orthogonal. They are not even real.

Because of the non-symmetry of A, we must talk about the right and left eigen values of A.

- **A right eigen pair of A is a solution (λ, V) that satisfies** $AV = \lambda V$
- **A left eigen pair of A is a solution (λ, W) that satisfies** $W^H A = \lambda W^H$ **where $(\ )^H$ is the complex conjugate transpose of its arguement. Note that the left eigenvectors of A are the right eigenvectors of A^T. Please verify this.**

Several Useful Observations

- **If (λ, V) is a right eigen pair of A, then so is (λ^*, V^*). Try it! Lets order these eigen solutions by absolute value of the imaginary part (the frequency) placing .**
- **Similarly, if (λ, W) is a left eigen pair of A, then so is (λ^*, W^*).**
- **If λ is a right eigen value of A, then it is also a left eigen value of A. The arguement is as follows. If $(A - \lambda I)V = 0$ for a non-trivial V then $A - \lambda I$ is singular, which implies not only that the columns of $A - \lambda I$ are linearly dependent, but that the rows of $A - \lambda I$ are also linearly dependent. The linear dependence of the rows of $A - \lambda I$ means there is a linear combination of them which is also zero: $W^H(A - \lambda I) = 0$. Lets order these eigen solutions identically to the corresponding right eigen solutions.**

Bi-Orthogoality

Though the right eigen vectors are not mutually orthogonal, the left eigenvectors are also not mutually orthogonal, there is an orthogonallity between members of the two sets.

For the sake of simplicity, for the moment, we assume that there are no repeated eigenvalues.

Consider one right eigen solution $(A - \lambda_n I) V_n = 0$**, and contract it by** W_m^H**:**

$$W_m^H A V_n = \lambda_n W_m^H V_n$$

Similarly, we consider the m'th left eigen solution and contract it by V_n**:**

$$W_m^H A V_n = \lambda_m W_m^H V_n$$

Bi-Orthogoality

Subtracting these two, we have $0 = (\lambda_n - \lambda_m) W_m^H V_n$

Since we had assumed that there are no repeated eigenvalues, we conclude that $W_m^H V_n = 0$ **and** $W_m^H A V_n$ **for** $n \neq m$**.**

If there are repeat eigenvalues, we select linear combinations of the corresponding eigenvectors to achieve mutual bi-orthogonality among those eigenvectors.

We normalize the V_n **so that** $W_n^H V_n = 1$ **and** $W_n^H A V_n = \lambda_n$**.**

Solution of Governing Equations

We postulate solutions $y = \sum_{k=1}^{2n} a_k(t) Y_k$ **to the governing equation**

$\dot{y} - Ay = \tilde{F}$ **and contract by** W_m^H **to obtain**

$$\dot{a}_m(t) - \lambda_m a_m(t) = W_m^H \tilde{F}(t)$$

In principle this could be solved by Laplace transforms:

$$\mathcal{L}\{a_m\} = \frac{W_m^H \mathcal{L}\{\tilde{F}\}}{s - \lambda_m} + \frac{a_m(0)}{s - \lambda_m}.$$

Note that the initial values are found from orthogonality

$$a_m(0) = W_m^H y(0).$$

Solution of Governing Equations

Lets let $(\lambda_{m'}, V_{m'})$ be the eigen pair conjugate to (λ_m, V_m) :

$$\lambda_{m'} = \lambda_m^* \text{ and } V_{m'} = V_m^*$$

The modal coordinates associated with $(\lambda_{m'}, V_{m'})$ are also solved by Laplace transforms:

$$\mathcal{L}\{a_{m'}\} = \frac{W_{m'}^H \mathcal{L}\{\tilde{F}\}}{s - \lambda_{m'}} + \frac{W_{m'}^H y(0)}{s - \lambda_{m'}}$$

$$= \frac{(W_m^H)^* \mathcal{L}\{\tilde{F}\}}{s - \lambda_m^*} + \frac{(W_m^H)^* y(0)}{s - \lambda_m^*}$$

Solution of Governing Equations

Lets add the contributions of each of these terms to the solution

$$\mathcal{L}\{a_m V_m + a_{m'} V_{m'}\} =$$

$$= \left[\left(\frac{V_m W_m^H}{s-\lambda_m}\right) + \left(\frac{V_m W_m^H}{s-\lambda_{m'}}\right)^*\right](\mathcal{L}\{\tilde{F}\} + y(0))$$

Note that the right hand side of the above equation is real. The complex modes contribute to the solution pair-wise as real values.

We reclaim the solution to the physical problem from $\begin{bmatrix} \dot{x} \\ x \end{bmatrix} = y$.