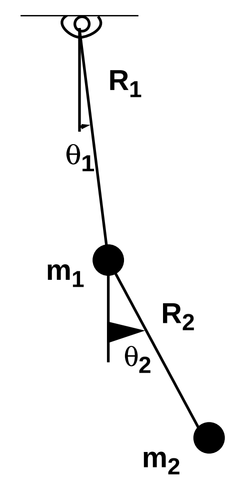
Slides of Lecture 8

Today's Class:

Review Of Homework From Lecture 7
Hamilton's Principle
More Examples Of Generalized Coordinates
Calculating Generalized Forces Via Virtual Work

Homework from Lecture 7

In the most recent class, we derived the governing equations for the compound pendulum. The homework assignment was to verify the derivation and to linearize the resulting equations.



We found the governing equations to be

$$(m_1 + m_2)R_1^2\ddot{\theta}_1 + m_2 R_1 R_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) - m_2 R_1 R_2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) + g(m_1 + m_2)R_1 \sin\theta_1 = 0$$

and

$$m_2 R_2^2 \ddot{\theta}_2 + m_2 R_1 R_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) + m_2 R_1 R_2 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) + g m_2 R_2 \sin\theta_2 = 0$$

Homework from Lecture 7

1. **Establish the location of stable equilibrium about which to linearize. By observation, that location is $\theta_1 = \theta_2 = 0$.**

2. **Substitute $\theta_1 = \cancelto{0}{\theta_1^s} + \Delta\theta_1$ and $\theta_2 = \cancelto{0}{\theta_2^s} + \Delta\theta_2$ into the governing equations and expand with Taylor series.**

$$(m_1 + m_2)R_1^2\ddot{\theta}_1 + m_2R_1R_2\ddot{\theta}_2\cancelto{1}{\cos(\theta_2 - \theta_1)}$$
$$-m_2R_1R_2\dot{\theta}_2^2\cancelto{\theta_2-\theta_1}{\sin(\theta_2 - \theta_1)} + g(m_1 + m_2)R_1\cancelto{\theta_1}{\sin\theta_1} = 0$$

$$m_2R_2^2\ddot{\theta}_2 + m_2R_1R_2\ddot{\theta}_1\cancelto{1}{\cos(\theta_2 - \theta_1)}$$

and

$$+ m_2R_1R_2\dot{\theta}_1^2\cancelto{\theta_2-\theta_1}{\sin(\theta_2 - \theta_1)} + gm_2R_2\cancelto{\theta_2}{\sin\theta_2} = 0$$

Homework from Lecture 7

3. Delete all terms involving powers and products of θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$

$$(m_1 + m_2)R_1^2\ddot{\theta}_1 + m_2R_1R_2\ddot{\theta}_2$$

$$-m_2R_1R_2\cancelto{0}{\dot{\theta}_2^2}(\theta_2 - \theta_1) + g(m_1 + m_2)R_1\theta_1 = 0$$

and

$$m_2R_2^2\ddot{\theta}_2 + m_2R_1R_2\ddot{\theta}_1$$

$$+ m_2R_1R_2\cancelto{0}{\dot{\theta}_1^2}(\theta_2 - \theta_1) + gm_2R_2\theta_2 = 0$$

Homework from Lecture 7

4. **Group terms to form the mass and stiffness matrix.**

$$\left.\begin{matrix} m_1 + m_2)R_1^2 & m_2 R_1 R_2 \\ m_2 R_1 R_2 & m_2 R_2^2 \end{matrix}\right] \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} g(m_1 + m_2)R_1 & 0 \\ 0 & g m_2 R_2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \left[\right.$$

Note that both the mass and stiffness matrices are symmetric and positive-definite.

The mass equation will be positive so long as we avoid massless degrees of freedom, that is we always want to choose our degrees of freedom so that $\dfrac{\partial T}{\partial \dot{q}_r} \neq 0$**.**

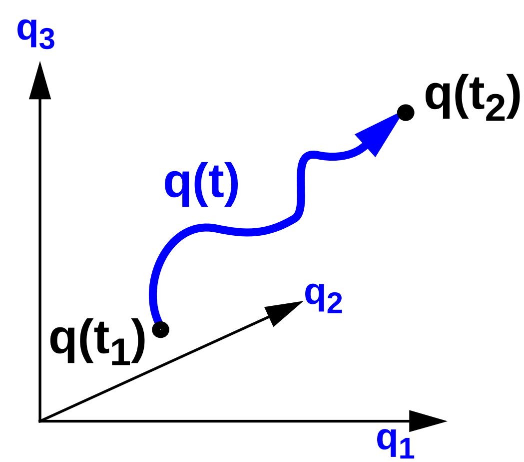
The stiffness matrix will always be non-negative definite. If there are no rigid-body modes, it will be positive definite. We shall discuss this more later.

Hamilton's Principle

We consider a mechanical system whose configuration at any time is characterized by the N generalized coordinates $\{q\}$. The system is subject to potential energy V and additional forces $\{F_r^A\}$ and evolves over the interval (t_1, t_2) according to the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} = F_r^A \text{ for each } r.$$

We can imagine the evolution of the system configuration over that interval by picturing the motion of a point whose coordinates are $\{q(t)\}$ in an N-dimensional Cartesian system.



Hamilton’s Principle

We contract these scalar equations with test functions η_r which are as yet undetermined except for the conditions $\eta_r(t_1) = \eta_r(t_2) = 0$ for each r, and then sum them.

$$\sum_{r=1}^{N} \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} + \frac{\partial V}{\partial q_r} - F_r^A \right] \eta_r(t)\, dt = 0$$

Hamilton's Principle

Integration by parts yields

$$\sum_{r=1}^{N}\int_{t_1}^{t_2}\left[-\frac{\partial T}{\partial q_r}+\frac{\partial V}{\partial q_r}-F_r^A\right]\eta_r(t)dt$$

$$+\sum_{r=1}^{N}\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\left(\frac{\partial T}{\partial \dot{q}_r}\right)\eta_r\right)-\frac{\partial T}{\partial \dot{q}_r}\dot{\eta}_r\right]dt = 0$$

Recalling that $\eta_r(t_1) = \eta_r(t_2) = 0$, the above integral simplifies

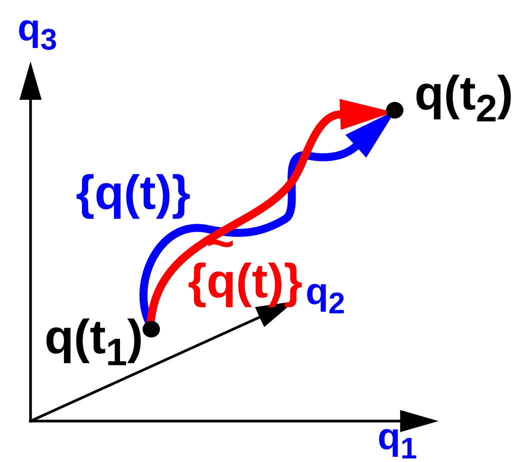
$$\sum_{r=1}^{N}\int_{t_1}^{t_2}\left\{\left[-\frac{\partial T}{\partial q_r}+\frac{\partial V}{\partial q_r}-F_r^A\right]\eta_r(t)-\frac{\partial T}{\partial \dot{q}_r}\dot{\eta}_r\right\}dt = 0$$

Hamilton's Principle

The above is true for all test functions η_r.

Let
$$\eta_r(t) = \tilde{q}_r(t) - q_r(t)$$
$$= \delta q_r(t)$$

where $\{\tilde{q}(t)\}$ is another path from $\{q(t_1)\}$ to $\{q(t_2)\}$ "near" the path taken by $\{q(t)\}$.



Our integrals can now be written:

$$\sum_{r=1}^{N} \int_{t_1}^{t_2} \left\{ -\left[\left(\frac{\partial T}{\partial q_r}\delta q_r + \frac{\partial T}{\partial \dot{q}_r}\delta \dot{q}_r\right)\right] + \left[\frac{\partial V}{\partial q_r} - F_r^A\right]\delta q_r \right\} dt = 0$$

Observe that the terms involving potential energy are a complete differential.

Hamilton's Principle

Rearranging the above:

$$\int_{t_1}^{t_2}\left(\sum_{r=1}^{N}\left\{\left[\left(\frac{\partial T}{\partial q_r}\delta q_r + \frac{\partial T}{\partial \dot{q}_r}\delta \dot{q}_r\right)\right] + \left[-\frac{\partial V}{\partial q_r} + F_r^A\right]\delta q_r\right\}\right)dt$$

$$= \int_{t_1}^{t_2}(\delta T - \delta V + \delta W)dt = 0$$

Where $\delta W(t) = F_r^A(t)(\tilde{q}_r(t) - q_r(t))$**.**

This form of Hamilton's principle asserts that the actual path is one

about which $\int_{t_1}^{t_2}(\delta T - \delta V + \delta W)dt = 0$

Hamilton’s Principle: Special Case

For the special case where the generalized forces F_r^A are prescribed loads, we can define the “Potential Energy of Loading”

$$A = -\sum_{r=1}^{N} F_r^A q_r$$

and the “Total Potential Energy” is $V^T = V + A$.

In this case, Hamilton’s Principle becomes: The true path in configuration space of the system makes the quantity

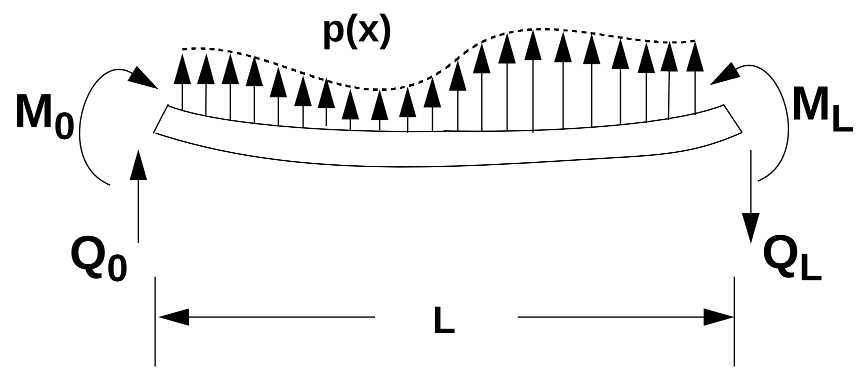
$$J = \int_{t_1}^{t_2} (T - V^T)\,dt$$ **stationary.**

Hamilton's Principle: Example Beam Bending Equation

The strain energy in an Euler-Bernoulli beam is

$$V = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx$$

. The potential energy of loading is



$$A = M_0 \frac{\partial y}{\partial x}\bigg|_0 - M_L \frac{\partial y}{\partial x}\bigg|_L - Q_0 y(0) + Q_L y(L) - \int_0^L p(x) y(x) dx$$

And the Kinetic Energy is $T = \frac{1}{2}\int_0^L m\left(\frac{\partial y}{\partial t}\right)^2 dx$ **where** m **is the mass per unit length of the beam.**

Hamilton's Principle: Example
Beam Bending Equation

Lets evaluate the virtual quantities, beginning with Kinetic Energy:

$$\delta T = \frac{1}{2}\int_o^L m\left[\frac{\partial}{\partial t}(y+\delta y)\right]^2 dx - \frac{1}{2}\int_0^L m\left(\frac{\partial y}{\partial t}\right)^2 dx$$

$$\cong \int_0^L m\left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial}{\partial t}\delta y\right)dx$$

and

$$\int_{t_1}^{t_2}\delta T\,dt = \int_{t_2}^{t_2}\left(\int_0^L m\left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial}{\partial t}\delta y\right)dx\right)dt$$

$$= \int_0^L m\left(\int_{t_1}^{t_2}\left[\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\delta y\right) - \ddot{y}\delta y\right]dt\right)dx$$

Hamilton's Principle: Example
Beam Bending Equation

$$\int_{t_1}^{t_2} \delta T\, dt = \cancelto{0}{\int_0^L m[\dot{y}\delta y]\big|_{t_1}^{t_2} dx} - \int_{t_1}^{t_2}\int_0^L m\left[\frac{\partial^2 y}{\partial t^2}\delta y\right] dx\, dt$$

Strain Energy:

$$\delta V = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2}{\partial x^2}(y+\delta y)\right)^2 dx - \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx$$

$$\cong \int_0^L EI y''\left(\frac{\partial^2}{\partial x^2}\delta y\right) dx$$

$$= EI\left(y''\frac{\partial}{\partial x}\delta y\right)\Bigg|_0^L - EI(y'''\delta y)\big|_0^L + \int_0^L EI y^{IV}\delta y\, dx$$

Hamilton's Principle: Example
Beam Bending Equation

The strain energy term becomes:

$$\int_{t_1}^{t_2} \delta V dt =$$

$$\int_{t_1}^{t_2} \left[EI\left(y'' \frac{\partial}{\partial x} \delta y \right)\Bigg|_0^L - EI(y''' \delta y)\big|_0^L + \int_0^L EI y^{IV} \delta y dx \right] dt$$

Hamilton’s Principle: Example
Beam Bending Equation

In a similar manner, we find the contribution from Potential Energy of Loading

$$\int_{t_1}^{t_2} \delta A dt$$

$$= \int_{t_1}^{t_2} \left[M_0 \frac{\partial}{\partial x} \delta y \Big|_0 - M_L \frac{\partial}{\partial x} \delta y \Big|_L - Q_0 \delta y(0) + Q_L \delta y(L) \right] dt$$

$$- \int_{t_1}^{t_2} \left[\int_0^L p(x) \delta y dx \right] dt$$

Hamilton's Principle: Example Beam Bending Equation

We can group terms.

We start with the terms involving $\delta y(0)$, $\delta y(L)$,

$\frac{\partial}{\partial x}\delta y|_0$, **and** $\frac{\partial}{\partial x}\delta y|_L$

$$\int_{t_1}^{t_2}\left[M_0\frac{\partial}{\partial x}\delta y|_0 - M_L\frac{\partial}{\partial x}\delta y|_L - Q_0\delta y(0) + Q_L\delta y(L)\right]dt$$

$$+\int_{t_1}^{t_2}\left[EI\left(y''\frac{\partial}{\partial x}\delta y\right)\Big|_0^L - EI(y'''\delta y)|_0^L\right]dt = 0$$

Hamilton's Principle: Example
Beam Bending Equation

From which we deduce $[Q_0 - EIy'''(0)]\delta y(0) = 0$**.**

A geometric boundary condition specifying $y(0)$ **implies that** $\delta y(0) = 0$**. If displacement is not specified there, then** $Q_0 = EIy'''(0)$**. This is a "natural" boundary condition.**

Similar interpretations are made of $[Q_L - EIy'''(L)]\delta y(L) = 0$**,**

$[M_0 - EIy''(0)]\frac{\partial}{\partial x}\delta y|_0 = 0$ **and** $[M_L - EIy''(L)]\frac{\partial}{\partial x}\delta y|_L = 0$**,**

Hamilton’s Principle: Example
Beam Bending Equation

Matching terms in the spacial integral we have

$$\int_{t_1}^{t_2}\int_0^L [m\ddot{y} + EIy^{IV} - p]\delta y(x,t)\,dx\,dt = 0$$

from which we conclude that

$$m\ddot{y} + EIy^{IV} = p(x,t)$$

Hamilton’s Principle

Hamilton’s principle is general and always works, though sometimes it is hard to evaluate.

In particular, note how Hamilton’s Principle is used to derive the partial differential governing equations.

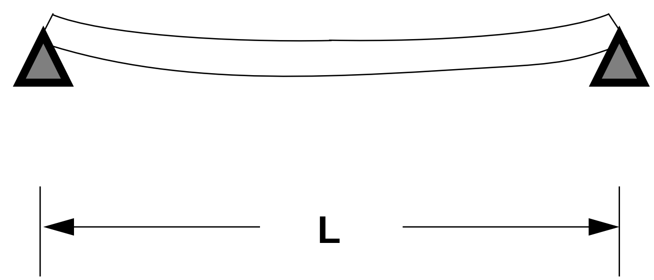
Also, we saw how to define the potential energy of loading and to use that with Hamilton’s principle. We will see that we can also use it in with Lagrange’s equations.

Change of Subject.

The following is an introduction to the method of ASSUMED MODES.

More On Generalized Degrees of Freedom Distributed Displacement

Lets consider an Euler Bernoulli beam simply supported at each end.



Initially, we assume that all forces are conservative. We postulate a displacement distribution of the sort

$$y(x,t) = A_1(t)\frac{x(L-x)}{L^2} + A_2(t)\frac{x^2(L-x)}{L^3}$$

$$= A_1(t)f_1(x) + A_2(t)f_2(x)$$

We shall derive Lagrange equations for the evolution of $A_1(t)$ and $A_2(t)$. These are our generalized degrees of freedom.

More On Generalized Degrees of Freedom Distributed Displacement

Kinetic Energy:

$$T = \frac{1}{2}\int_0^L m\dot{y}^2 dx = \frac{1}{2}\int_0^L m[\dot{A}_1(t)f_1(x) + \dot{A}_2(t)f_2(x)]^2 dx$$

$$= \frac{1}{2}[(\dot{A}_1)^2 I_1 + 2\dot{A}_1\dot{A}_2 I_2 + (\dot{A}_2)^2 I_3]$$

where $I_1 = \int_0^L m f_1(x)^2 dx = \frac{mL}{30}, I_2 = \int_0^L m f_1(x) f_2 x dx = \frac{mL}{60}$

and $I_3 = \int_0^L m f_2(x)^2 dx = \frac{mL}{105}$

More On Generalized Degrees of Freedom Distributed Displacement

Strain Energy:

$$V = \frac{1}{2}\int_0^L EI(y'')^2 dx = \frac{1}{2}\int_0^L EI[A_1(t) f_1''(x) + A_2(t) f_2''(x)]^2 dx$$

$$= \frac{1}{2}[(A_1)^2 I_4 + 2A_1 A_2 I_5 + (A_2)^2 I_6]$$

where

$$I_4 = \int_0^L EI f_1''(x)^2 dx = \frac{EI4}{L^3},\ I_6 = \int_0^L EI f_2''(x)^2 dx = \frac{EI4}{L^3}$$ **and**

$$I_5 = \int_0^L EI f_1''(x) f_2''(x) dx = \frac{EI2}{L^3}$$

More On Generalized Degrees of Freedom Distributed Displacement

Lagrange Equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{A}_1}\right) - \frac{\partial T}{\partial A_1} + \frac{\partial V}{\partial A_1} = \ddot{A}_1 I_1 + \ddot{A}_2 I_2 + A_1 I_4 + A_2 I_5 = 0$$

and

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{A}_2}\right) - \frac{\partial T}{\partial A_2} + \frac{\partial V}{\partial A_2} = \ddot{A}_1 I_2 + \ddot{A}_2 I_3 + A_1 I_5 + A_2 I_6 = 0$$

In matrix form: $\frac{mL}{15}\begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} \ddot{A}_1 \\ \ddot{A}_2 \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note that both matrices are symmetric, positive definite.

Generalized Forces, Calculated by Method of Virtual Work

Recall that the generalized force associated with the generalized coordinate q_r is $F_r = \sum_n \vec{F}_n \cdot \frac{\partial \vec{x}_n}{\partial q_r}$

We examine the incremental work associated with increments of q_r:

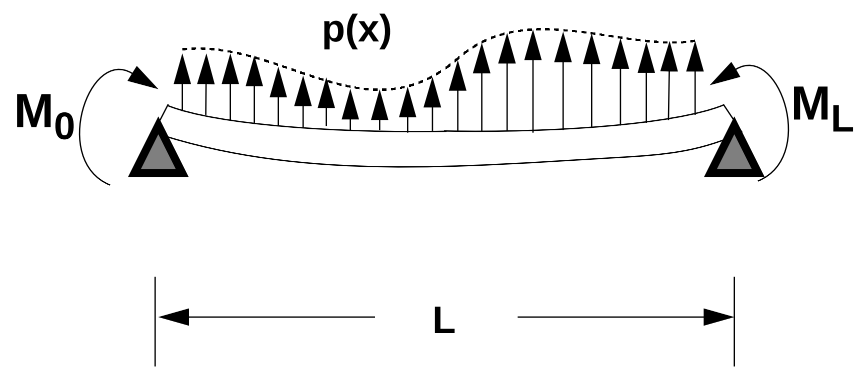
$$\delta W = F_r \delta q_r = \sum_n \vec{F}_n \cdot \frac{\partial \vec{x}_n}{\partial q_r} \delta q_r = \sum_n \vec{F}_n \cdot \delta \vec{x}_n$$

The generalized force associated with the generalized coordinate q_r is

$$F_r = \frac{\delta W}{\delta q_r}$$

Generalized Forces, Calculated by Method of Virtual Work

Lets consider an Euler Bernoulli beam simply supported at each end.



We consider moments M_0 and M_L applied at the ends and a distributed traction applied along the length of the beam.

We postulate a displacement distribution of the sort

$$y(x, t) = A_1(t)\frac{x(L-x)}{L^2} + A_2(t)\frac{x^2(L-x)}{L^3}$$

$$= A_1(t) f_1(x) + A_2(t) f_2(x)$$

Lets calculate the generalized forces associated with the generalized coordinates A_1 and A_2.

Generalized Forces, Calculated by Method of Virtual Work

The work done to the structure by the forces acting through a virtual displacement $\delta y = (\delta A_1) f_1(x)$, is

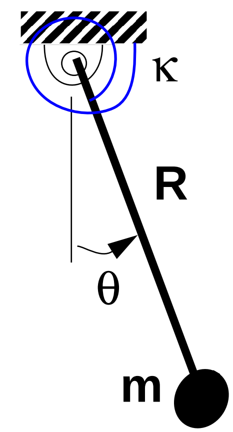
$$\delta W = \delta A_1 \left(-M_0 f_1'(0) + M_L f_1'(L) + \int_0^L p(x) f_1(x) dx \right)$$

so $$F_{A_1} = \left(-M_0 f_1'(0) + M_L f_1'(L) + \int_0^L p(x) f_1(x) dx \right)$$

F_{A_2} **can be calculated similarly.**

Homework for Lecture 8
A numerical experiment with linearization

Many times we have derived the equations for a spring reinforced pendulum: $\ddot{\theta} + \left(\frac{\kappa}{mR^2}\theta + \frac{g}{R}\sin\theta\right) = 0$.



The linearized form is $\ddot{\theta} + \omega_L^2\theta = 0$ where

$$\omega_L^2 = \frac{\kappa}{mR^2} + \frac{g}{R}.$$

We use the linearized frequency to non-dimensionalize the time parameter. Define $\tau = \omega_L t$, define $\phi(\tau) = \theta(\tau/\omega_L)$, and define

$$\alpha = \frac{g/R}{\omega_L^2}$$

Homework for Lecture 8 continued

Then $\frac{d^2\phi}{d\tau^2} = \frac{d^2\theta}{dt^2}\frac{1}{\omega_L^2}$ **and** $\frac{d^2\phi}{d\tau^2} + [(1-\alpha)\phi + \alpha\sin\phi] = 0$

1. Solve numerically the dimensionless governing equation for the initial conditions: $\phi(0) = \pi$ **and** $\left.\frac{d\phi}{d\tau}\right|_0 = 0$ **over the period** $(0, 6\pi)$ **for the three cases:** $\alpha = 0$, $\alpha = 1/2$, **and** $\alpha = 1$.

2. Do the same as above but for the initial conditions $\phi(0) = \frac{\pi}{6}$ **and** $\left.\frac{d\phi}{d\tau}\right|_0 = 0$

3. Compare and discuss the your results for parts 1 and 2.

Next Time

Example problems students choice and discussion of past material.

Discussion of mid-term exam.