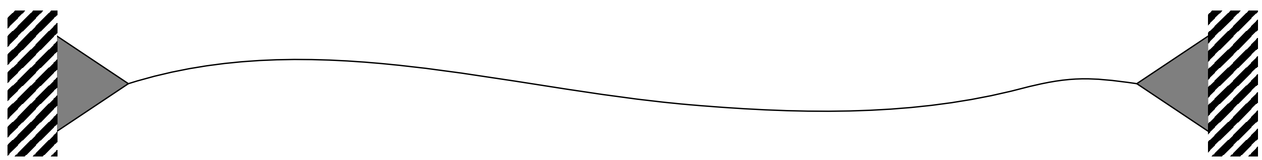
Lecture 16

Over the next few weeks, we shall address the vibration of continuous structure.

- **We shall begin deriving the governing equations for idealized structures such as strings, rods, Euler-Bernoulli beams, Timoshenko beams, and plates. We shall do this exploiting D'Alembert's principle and Hamilton's principle.**
- **Where possible, we shall derive closed form solutions to dynamics of these structures in terms of eigenmodes.**
- **We shall explore approximate solutions to these structures.**
- **If time permits, we shall begin an examination of waves.**

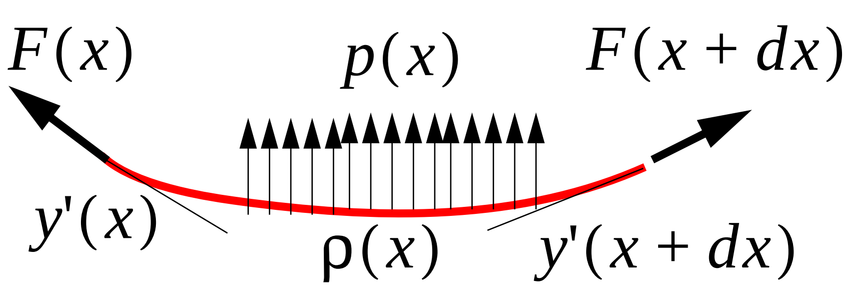
Strings

Strings are a good place to begin the process of deriving governing equations and closed-form solutions.



Strings are one-dimensional structures too thin to exhibit bending stiffness. Tension in the string serves to drive it toward the equilibrium configuration - a straight line.

Lets draw a free-body diagram and derive governing equations.

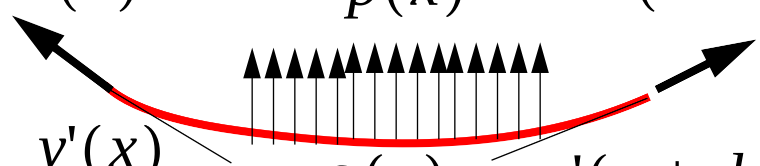


The tension on the string is $F(x)$, the density of the string is $\rho(x)$, and the distributed load is $p(x)$

Strings

We balance the vertical component of force to find

$$\rho(x)\ddot{y}(x) = (F(x)y'(x))' + p(x,t)$$



In what follows, we shall assume that $F(x) = F = \text{Constant}$ **& that** $\rho(x) = \rho = \text{Constant}$.

We choose easy boundary conditions: $y(0,t) = y(L,t) = 0$.

Lets now solve the unforced initial value problem:

$$\rho\ddot{y}(x) = (Fy'(x))'$$

subject to $y(x,0) = y_0(x)$ **and** $\dot{y}(x,0) = v_0(x)$

Strings

Lets solve $\rho \ddot{y}(x) = (F y'(x))'$ **by separation of variables**

$$y(x, t) = \sum_n X_n(x) T_n(t)$$

This yields $\dfrac{\ddot{T}_n}{T_n} = \dfrac{F}{\rho} \dfrac{X_n''}{X_n} = -\lambda_n^{\,2}$ **for each** n**.**

Solving $X_n'' + \lambda_n^2 \dfrac{\rho}{F} X_n = 0$**, subject to** $X_n(0) = X_n(L) = 0$**, we find that**

$$X_n = \sin \frac{n\pi x}{L} \text{ and } \left(\frac{n\pi}{L}\right)^2 = \lambda_n^2 \frac{\rho}{F}$$

Strings

$$\left(\frac{n\pi}{L}\right)^2 = \lambda_n^2\frac{\rho}{F} \Rightarrow \lambda_n = \pm\frac{n\pi}{L}\left(\frac{F}{\rho}\right)^{1/2}.$$

The temporal part of the solution is the solution to $\ddot{T}_n + \lambda_n^2 T_n = 0$

which has solutions $T_n(t) = A_n\cos(\lambda_n t) + B_n\sin(\lambda_n t)$

The solution to the problem is

$$y(x,t) = \sum_n (A_n\cos(\lambda_n t) + B_n\sin(\lambda_n t))\sin\frac{n\pi x}{L}$$

The velocity of the string is

$$\dot{y}(x,t) = \sum_n \lambda_n(-A_n\sin(\lambda_n t) + B_n\cos(\lambda_n t))\sin\frac{n\pi x}{L}$$

Strings

Lets match initial conditions.

From prescribed initial position, $y_0(x) = \sum_n A_n \sin\frac{n\pi x}{L}$

From prescribed initial velocity, $v_0(x) = \sum_n \lambda_n B_n \sin\frac{n\pi x}{L}$.

We use these to solve for the coefficients by exploiting the orthogonality of sine and cosine.

$$A_n = \frac{2}{L}\int_0^L y_0(x)\sin\frac{n\pi x}{L}dx \text{ \& } B_n = \frac{2}{\lambda_n L}\int_0^L v_0(x)\sin\frac{n\pi x}{L}dx$$

Please confirm these. Which is the fundamental tone?

Strings and Positive Operators

Note that we could have posed this in terms of kinetic energy and potential energy operators.

Let K **be the differential operator** $K[y] = -\frac{\partial}{\partial x}\left(F\frac{\partial}{\partial x}(y)\right)$ **and we consider its operation only on the linear space of smooth functions that satisfy the geometric boundary conditions (**$y(0, t) = y(L, t) = 0$**).**

K **is a symmetric (self adjoint) operator is that** $(y_1, Ky_2) = (K(y_1), y_2)$ **where** (a, b) **is the inner product of functions** a **and** b**. In the case of the string,** $(a, b) = \int_0^L a(x)b(x)dx$**.**

Strings and Positive Operators

We prove that K is a self-adjoint operator directly.

$$(y_1, Ky_2) = \int_0^L y_1(x)\left(-\frac{\partial}{\partial x}\left(T\frac{\partial}{\partial x}(y_2)\right)\right)dx$$

$$= \int_0^L \left[-\frac{\partial}{\partial x}\left(y_1(x)T\frac{\partial}{\partial x}(y_2)\right) + T\frac{\partial y_1}{\partial x}\frac{\partial y_2}{\partial x}\right]dx$$

$$= \underbrace{\left[y_1(x)T\frac{\partial}{\partial x}(y_2)\right]_0^L}_{0} + \int_0^L T\frac{\partial y_1}{\partial x}\frac{\partial y_2}{\partial x}dx$$

$$= \int_0^L T\frac{\partial y_2}{\partial x}\frac{\partial y_1}{\partial x}dx = (Ky_1, y_2)$$

Strings and Positive Operators

It is also easily seen that K is a positive definite operator

$$(y, Ky) = \int_0^L T \frac{\partial y}{\partial x}\frac{\partial y}{\partial x} dx > 0$$ for all non-trivial $y(x)$.

We define the mass operator $My(x) = \rho y(x)$.

It is obvious that M is symmetric: $(y_1, My_2) = (My_1, y_2)$,

and that M is positive definite: $(y, My) > 0$ for all non-trivial $y(x)$.

The governing equation now has the form $M\ddot{y} + Ky = 0$. Except for a sign difference, this looks similar to the matrix equations for discrete systems.

Strings and Positive Operators
Eigenmodes For Continuous Systems

Lets define U_0 to be the set of twice differentiable functions X defined in $[0, L]$ that satisfy the boundary conditions at 0 and L.

We consider the functional $\lambda(X)$: $\lambda^2(X) = \dfrac{(X, KX)}{(X, MX)}$ operating on functions in $X \in U_0$,

Let $\lambda_1^2 = \min\limits_{X \in U_0} \lambda^2(X)$ and let X_1 be the function that achieves that minimum.

Let $U_1 = \{X \in U_0 : (X, MX_1) = 0\}$ **and let** $\lambda_2^2 = \min_{X \in U_1} \lambda^2(X)$

and let X_2 **be the function that achieves that minimum.**

One continues in this manner:

defining $U_n = \{X \in U_{n-1} : (X, MX_n) = 0\}$ **and letting**

$\lambda_{n+1}^2 = \min_{X \in U_n} \lambda^2(X)$ **and** X_{n+1} **be the function that achieves that minimum.**

Lets examine the character of the solutions to this optimization problem.

Examine $\frac{d}{d\alpha}\frac{((X_1 + \alpha\tilde{X}), K(X_1 + \alpha\tilde{X}))}{((X_1 + \alpha\tilde{X}), M(X_1 + \alpha\tilde{X}))}\bigg|_{\alpha = 0} = 0$ **where** $\tilde{X}(x)$ **is any function of** x **satisfying the boundary conditions.**

Appropriate differentiation shows that $(\tilde{X}, KX_1 - \lambda_1^2 MX_1) = 0$ **for all** $\tilde{X}$**. This implies that** $KX_1 - \lambda_1^2 MX_1 = 0$**.** (λ_1, X_1) **are and eigen pair for the governing equation.**

Strings and Positive Operators
Eigenmodes For Continuous Systems

Similarly, Examine $\left.\dfrac{d}{d\alpha}\dfrac{((X_n + \alpha\tilde{X}), K(X_n + \alpha\tilde{X}))}{((X_n + \alpha\tilde{X}), M(X_n + \alpha\tilde{X}))}\right|_{\alpha = 0} = 0$ where

$\tilde{X}(x)$ is any function of x satisfying the boundary conditions and orthogonal to the previous eigen functions.

Appropriate differentiation shows that $(\tilde{X}, KX_n - \lambda_n^2 MX_n) = 0$ for all $\tilde{X}$ orthogonal to the previous eigen-functions. This implies that $KX_n - \lambda_n^2 MX_n = 0$. (λ_n, X_n) are and eigen pair for the governing equation.

Strings and Positive Operators
Orthogonality of Eigen function

We examine orthogonality in the same way that we did for discrete systems.

Say that (λ_m, X_m) **and** (λ_n, X_n) **are eigen pairs with distinct eigen values:** $\lambda_n \neq \lambda_m$**. We examine the following inner products:**

$$(X_m, KX_n) - \lambda_n^2 (X_m, MX_n) = 0$$ **and**

$$(X_n, KX_m) - \lambda_m^2 (X_n, MX_m) = 0.$$

Subtracting one from another, we find $(\lambda_n^2 - \lambda_m^2)(X_n, MX_m) = 0$**.**

Eigen functions of distinct eigenvalues are orthogonal w.r.t. M**.**

Strings and Positive Operators
Orthogonality of Eigen function

$(X_m, KX_n) - \lambda_n^2 (X_m, MX_n) = 0$ **and**

$$(X_n, KX_m) - \lambda_m^2 (X_n, MX_m) = 0$$

Similarly, if the eigenvalues are distinct, we multiply the first of the equations by λ_m^2 and the second by λ_n^2 and then subtract to find

$$(\lambda_n^2 - \lambda_m^2)(X_n, KX_m) = 0 .$$

Eigen functions of distinct eigenvalues are orthogonal w.r.t. K.

Strings and Positive Operators
Orthogonality of Eigen function

What happens if there are repeated eigenvalues.

- **The above proof shows that the corresponding eigen functions can be made orthogonal to eigen functions of all other eigenvalues**
- **The eigen functions of the repeated eigenvalues can be combined in such a way as to make them orthogonal. We shall see some examples with beam bending.**

Strings and Eigenvalues

By the way, we see that $X_n = \sin\frac{n\pi x}{L}$ **is the n'th eigen function of** $(K - \lambda^2 M)X = 0$**, where** $KX = -(FX')'$ **and** $MX = \rho X$ **and the corresponding eigen value is** $\lambda_n^2 = \frac{F}{\rho}\left(\frac{n\pi}{L}\right)^2$**.**

Note that frequency increases with tension, decreases with density, and decreases with length.

Strings and Energy

By inspection, the kinetic energy is

$$T(t) = \frac{\rho}{2}\int_0^L (\dot{y}(x,t))^2 dx = \frac{1}{2}(\dot{y}, M\dot{y}).$$

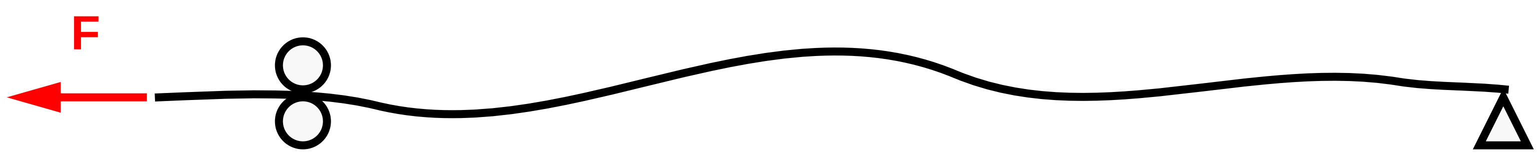
Lets also note that the potential energy is

$$V(t) = \frac{F}{2}\int_0^L (y'(x,t))^2 dx = \frac{1}{2}(y, Ky)$$

The equivalence of the above integral and inner product was established in slide Strings and Positive Operators on page 8. That this is the strain energy is shown next.

Strings and Energy

Lets consider the experiment where the string is pulled through a pulley at the boundary.



The distance that the string must be pulled through the pulleys in order to stretch the string into its deformed shape is the arc length of the curved string minus the distance between the boundaries

$$d = \int_0^L \sqrt{1 + y'^2}\, dx - L \cong \frac{1}{2}\int_0^L y'^2\, dx$$

and the work involved is $W = Fd = \frac{F}{2}\int_0^L y'^2\, dx$.

Strings and the Rayleigh Quotient

The quotient that we were examining before, $\lambda^2(X) = \dfrac{(X, KX)}{(X, MX)}$ **actually reflects the energy in periodic motion.**

Say that the deformation is $y(x, t) = Re\{e^{i\omega t}\}X(x)$**. Then** $\dot{y} = Re\{i\omega e^{i\omega t}\}X(x)$**. The maximum kinetic energy in a cycle is**

$$\max_{0 < t < \frac{2\pi}{\omega}} \int_0^L \frac{\rho}{2}\dot{y}(x, t)^2 dx = \max_{0 < t < \frac{2\pi}{\omega}} (Re\{i\omega e^{i\omega t}\})^2 \frac{1}{2}(X, MX)$$

$$= \frac{1}{2}\omega^2(X, MX)$$

Strings and the Rayleigh Quotient

Similarly, the maximum strain energy in each cycle is $\frac{1}{2}(X, KX)$.

We now see that $\lambda_n^2 = \omega_n^2 = \dfrac{(X, KX)}{(X, MX)}$ is the ratio of maximum strain energy in a mode to the maximum kinetic energy in that mode.

The function $\lambda^2(X) = \dfrac{(X, KX)}{(X, MX)}$ is called the Rayleigh quotient and is often used to obtain upper bounds for the first natural frequency.

Strings and the Rayleigh Quotient

Lets see how well $f(x) = x(L-x)$ approximates the first eigen function:

$$\lambda^2(f) = \frac{\int_0^L F(f'(x))^2 dx}{\int_0^L \rho(f(x))^2 dx} = \frac{FL^3/3}{\rho L^5/30} = 10\frac{F}{\rho L^2}$$

This is not to far from the exact solution

$$\lambda^2\left(\sin\frac{\pi x}{L}\right) = \frac{\pi^2 F}{\rho L^2} = 9.87\frac{F}{\rho L^2}$$

Derivation of String Equations via Hamilton's Principle

We derived the equation of motion for the string via a straightforward application of D'Alemberts principle (balancing force and inertial terms). Lets try it now with Hamilton's principle.

The Weak form of Hamilton's Principle: asserts that the actual path is one about which $\int_{t_1}^{t_2} (\delta T - \delta V + \delta W) dt = 0$**. Here,** δW **is the virtual work of external forces pushing the system from one path in configuration space to another.**

Lets first ignore all external forces.

Derivation of String Equations via Hamilton's Principle

$$0 = \int_{t_1}^{t_2} (\delta T - \delta V)\, dt$$

$$= \delta \int_{t_1}^{t_2} \frac{\rho}{2}\left(\int_0^L (\dot{y}(x,t))^2 dx \right) dt - \delta \int_{t_1}^{t_2} \frac{F}{2}\left(\int_0^L (y'(x,t))^2 dx \right) dt$$

$$= \int_0^L \int_{t_1}^{t_2} \left[\frac{\partial}{\partial t}(\rho \dot{y} \delta y) - \rho \ddot{y} \delta y \right] dt\, dx - \int_{t_1}^{t_2} \int_0^L F[(y'\delta y)' - y''\delta y]\, dx\, dt$$

$$= \int_0^L \overbrace{(\rho \dot{y} \delta y)\Big|_{t_1}^{t_2}}^{0} dx - \int_{t_1}^{t_2} \overbrace{(F(y'\delta y)\Big|_{t_1}^{t_2})}^{0} dt$$

$$- \int_0^L \left(\int_{t_1}^{t_2} [\rho \ddot{y} - F y'']\, \delta y\, dt \right) dx$$

Next Time

A little more on Strings

Rods