Define your strain!

Consider elongation of a bar from an initial length L_o to a final length L. The Seth-Hill family of strain measures is defined

$$\varepsilon = \frac{1}{\kappa}(\lambda^{\kappa} - 1), \text{ where } \lambda = \frac{L}{L_o} \tag{0.1}$$

Lengths are always positive, so we recognize that $0 < \lambda < \infty$. Some commonly used strain measures correspond to different choices of the parameter κ:

- Engineering strain ($\kappa = 1$): $\varepsilon^{\text{eng}} = \lambda - 1 = \dfrac{L - L_o}{L_o}$
- "True" strain ($\kappa = -1$): $\varepsilon^{\text{true}} = 1 - \dfrac{1}{\lambda} = \dfrac{L - L_o}{L}$
- Logarithmic strain ($\kappa \to 0$): $\varepsilon^{\log} = \ln\lambda = \ln\left(\dfrac{L}{L_o}\right)$
- Lagrange strain ($\kappa = 2$): $\varepsilon^{\text{Lag}} = \dfrac{1}{2}(\lambda^2 - 1)$

First of all, there is nothing innately better about any of these strain measures — they all legitimate quantify the deformation. If stress can be written in terms of one of these strains, then it can be written as a function of any of the *other* strain measures as well. Usually, a researcher will select the strain measure for which the stress-strain curve is most *linear*. Note that all of the strain measures are approximately equal in the limit of *infinitesimal* strains. However, there is a big difference between *infinitesimal* and *small*. As explained below, even a 1% strain makes the various strain measures agree only to a single significant digit.

Suppose that an experimentalist measures a stretch of $\lambda = 1.01000$, accurate to six significant digits. Then

$$\varepsilon^{\text{eng}} = 0.01000, \ \varepsilon^{\text{true}} = 0.00990, \ \varepsilon^{\log} = 0.00995, \ \varepsilon^{\text{Lag}} = 0.01005 \tag{0.2}$$

Even at such small strains, the definitions of strain differ by as much as 1.5 percent. If the experimentalist reports only a value of strain without defining the strain measure, then the result may be trusted only to *one significant digit* despite the fact that the experiment was accurate to six places! Failure to define strain results in loss of important data. Experimentalists and theorists alike must be very diligent about defining what *their* definition of strain is in *their* own work. Consider the relative discrepancy ξ between Lagrange strain and engineering strain:

$$\xi = \frac{\varepsilon^{\text{Lag}} - \varepsilon^{\text{eng}}}{\varepsilon^{\text{eng}}} = \frac{(1 + \varepsilon^{\text{eng}})^2 - 1}{2\varepsilon^{\text{eng}}} - 1 = \frac{\varepsilon^{\text{eng}}}{2} \tag{0.3}$$

Note that $d\xi / d\varepsilon^{\text{eng}} = 1/2$, which means that the discrepancy increases by half a percent for each percent of strain. The discrepancy between Lagrange strain and true strain is even higher. Namely, for each percent increase in strain, the discrepancy between the common strain measures increases by more than 1.5 percent!

Incidentally, for three-dimensional states of stress, the Seth-Hill strain measure generalizes to

$$\underset{\approx}{\varepsilon} = \frac{1}{\kappa}(\underset{\approx}{\boldsymbol{U}}^{\kappa} - \underset{\approx}{\boldsymbol{I}}) \quad , \tag{0.4}$$

where $\underset{\approx}{\boldsymbol{U}}$ is the stretch tensor from the polar decomposition of the deformation gradient tensor $\underset{\approx}{\boldsymbol{F}}$. That is, $\underset{\approx}{\boldsymbol{U}} = (\underset{\approx}{\boldsymbol{F}}^T \bullet \underset{\approx}{\boldsymbol{F}})^{1/2}$, and therefore $\underset{\approx}{\boldsymbol{U}}^{\kappa} = (\underset{\approx}{\boldsymbol{F}}^T \bullet \underset{\approx}{\boldsymbol{F}})^{\kappa/2}$. This operation is easy to compute if κ is an *even* integer, which explains the popularity of the Lagrange strain (for which $\kappa=2$). If κ is not an even integer, computing $\underset{\approx}{\boldsymbol{U}}^{\kappa}$ requires an expensive eigenvalue computation.