WORKSHOP:
Geometrical interpretation of
Radial and Oblique Return Methods

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ABSTRACT: Return algorithms are probably the most popular means of numerically solving conventional plasticity equations. The basic tenets of these techniques are here rigorously justified and interpreted geometrically in 6D stress space. For any return algorithm, the first step is to tentatively assume elastic behavior throughout a given time step. If the resulting "trial" stress is forbidden (i.e. if it violates the yield condition), then the tentative assumption of elastic response is rejected. Even when it is found to violate the yield condition, the trial stress is nevertheless useful because it can then be projected back to the plastic yield surface to give the updated stress. The return algorithm is called "normal" or "orthogonal" if the trial stress is projected directly to the nearest point on the yield surface. The return method is called "radial" or "Prandtl" when the projection is accomplished by reducing the magnitude of the trial stress deviator. Return algorithms are often wrongly regarded as numerical "tricks" because they appear to be ad hoc means of keeping the stress on the yield surface. It is natural to inquire whether other approaches might be more accurate for the same computational cost, but it is shown here that return methods are rigorously justifiable and appear to correspond to optimal numerical accuracy and efficiency. It is shown that issues such as plastic stability, dissipation, and convexity dictate appropriate choices for the quantities that are presumed known in the derivation of return algorithms; it is not the return algorithm per se that addresses such physical concerns. It is proved that the correct return direction is dictated by the governing equations and is not aligned with the plastic strain rate except under certain conditions. Consequently, normality of the plastic strain rate does not necessarily correspond to normality of the return direction, and vice versa. These claims are proved first in the context of stationary yield surfaces and then generalized to permit hardening or softening. The technical note is intended to provide nothing more than geometrical insight into known results.

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Tensors are vectors!

To a mathematician, a vector is a member of a set for which *addition* and *scalar multiplication* satisfy certain rules.

Many familiar 3D vector concepts and theorems also apply to tensors when regarded as 9D vectors.

3D vector operations	9D tensor operations
$\underset{\sim}{u} = \alpha \underset{\sim}{c}$ *means* $u_i = \alpha c_i$	$\underset{\approx}{U} = \alpha \underset{\approx}{C}$ *means* $U_{ij} = \alpha C_{ij}$
$\underset{\sim}{v} = \underset{\sim}{a} + \underset{\sim}{b}$ *means* $v_i = a_i + b_i$	$\underset{\approx}{V} = \underset{\approx}{A} + \underset{\approx}{B}$ *means* $V_{ij} = A_{ij} + B_{ij}$

3D inner product

$$\underset{\sim}{r} \bullet \underset{\sim}{s} \ \textit{means} \ \sum_{i=1}^{3} r_i s_i$$

9D inner product

$$\underset{\approx}{R} : \underset{\approx}{S} \ \textit{means} \ \sum_{i=1}^{3} \sum_{j=1}^{3} R_{ij} S_{ij}$$

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Projection operations

Orthogonal projection

Oblique projection

Plane perpendicular to $\underset{\sim}{n}$

Plane perpendicular to $\underset{\sim}{b}$
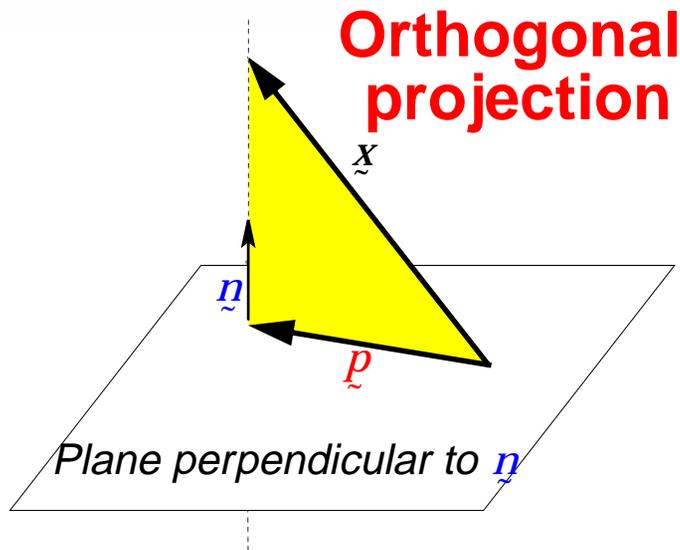
$$\underset{\sim}{p} = \underset{\sim}{x} - \underset{\sim}{n}(\underset{\sim}{n} \bullet \underset{\sim}{x})$$
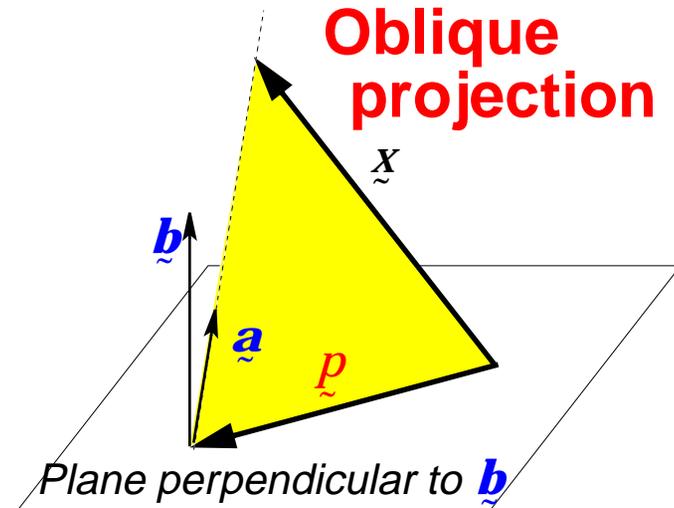
$$\underset{\sim}{p} = \underset{\sim}{x} - \frac{\underset{\sim}{a}(\underset{\sim}{b} \bullet \underset{\sim}{x})}{\underset{\sim}{a} \bullet \underset{\sim}{b}}$$

Note: $\underset{\sim}{b}$ defines the target plane; $\underset{\sim}{a}$ defines projection direction.

Analog for 9D tensor space:

$$P(\underset{\approx}{X}) = \underset{\approx}{X} - \frac{\underset{\approx}{A}(\underset{\approx}{B} : \underset{\approx}{X})}{\underset{\approx}{A} : \underset{\approx}{B}}$$

Projections are linear. . . $P(\alpha_1 \underset{\approx}{X}_1 + \alpha_2 \underset{\approx}{X}_2) = \alpha_1 P(\underset{\approx}{X}_1) + \alpha_2 P(\underset{\approx}{X}_2)$

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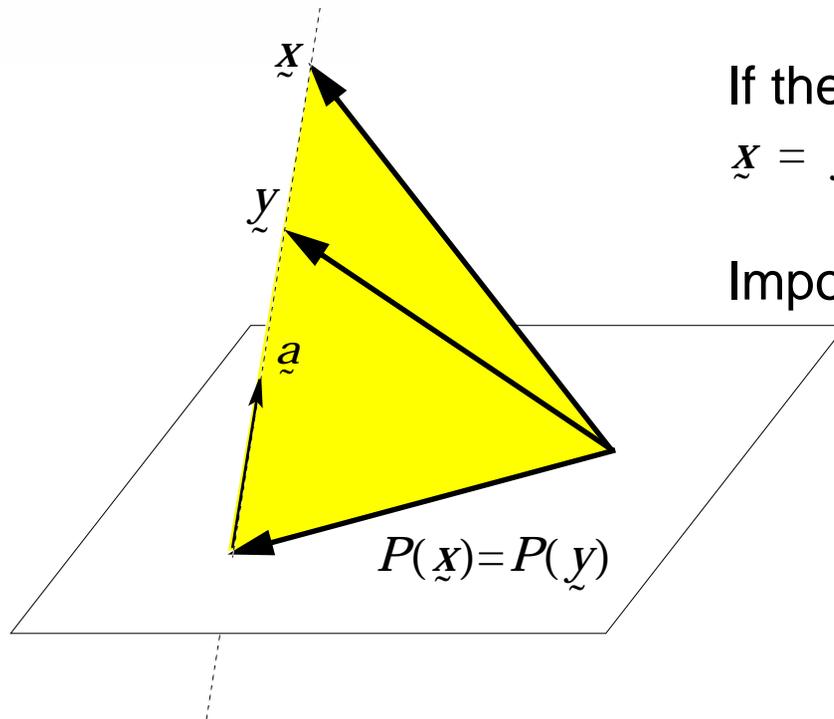
If there is a β such that

$$\underset{\sim}{x} = \underset{\sim}{y} + \beta \underset{\sim}{a}, \quad \text{then} \quad P(\underset{\sim}{x}) = P(\underset{\sim}{y}).$$

Important: *converse is true too!*

Analog for tensors:

If $\quad \underset{\approx}{X} = \underset{\approx}{Y} + \beta \underset{\approx}{A} \quad$ then $\quad P(\underset{\approx}{X}) = P(\underset{\approx}{Y}) \quad$ and vice versa.

Corollary: $P(P(\underset{\approx}{X})) = P(\underset{\approx}{X})$ (projecting twice makes no change).

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Nonhardening plasticity

Known:

$\underset{\approx}{B}$, gradient of yield function ($B_{ij} = \partial f / \partial \sigma_{ij}$).

$\underset{\approx}{\dot{\varepsilon}}$, total strain rate.

$\underset{\approx}{E}$, fourth-order elastic tangent stiffness tensor.

$\underset{\approx}{M}$, direction of the plastic strain rate.

Unknown:

$\underset{\approx}{\dot{\sigma}}$, rate of stress

$\underset{\approx}{\dot{\varepsilon}}^e$, elastic part of the strain rate

$\underset{\approx}{\dot{\varepsilon}}^p$, plastic part of the strain rate.

λ, magnitude of the plastic part of the strain rate.

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Governing equations

$\dot{\underset{\approx}{\varepsilon}} = \dot{\underset{\approx}{\varepsilon}}^e + \dot{\underset{\approx}{\varepsilon}}^p$ *strain rate decomposition*

$\dot{\underset{\approx}{\varepsilon}}^p = \dot{\lambda}\underset{\sim}{M}$ *plastic strain direction is known*

$\dot{\underset{\approx}{\sigma}} = \underset{\approx}{E}:\dot{\underset{\approx}{\varepsilon}}^e$ *stress rate linear in elastic strain rate*

$\underset{\approx}{B}:\dot{\underset{\approx}{\sigma}} = 0$ *stress stays on (nonhardening) yield surface*

Solution:

Note $\dot{\underset{\approx}{\varepsilon}}^e = \dot{\underset{\approx}{\varepsilon}} - \dot{\underset{\approx}{\varepsilon}}^p = \dot{\underset{\approx}{\varepsilon}} - \dot{\lambda}\underset{\sim}{M}$, so $\dot{\underset{\approx}{\sigma}} = \underset{\approx}{E}:(\dot{\underset{\approx}{\varepsilon}} - \dot{\lambda}\underset{\approx}{M})$. For convenience,

define $\dot{\underset{\approx}{\sigma}}^{\text{trial}} = \underset{\approx}{E}:\dot{\underset{\approx}{\varepsilon}}$ and $\underset{\approx}{A} = \underset{\approx}{E}:\underset{\approx}{M}$. Then $\dot{\underset{\approx}{\sigma}} = \dot{\underset{\approx}{\sigma}}^{\text{trial}} - \dot{\lambda}\underset{\approx}{A}$

Enforce last equation to get $\underset{\approx}{B}:[\dot{\underset{\approx}{\sigma}}^{\text{trial}} - \dot{\lambda}\underset{\approx}{A}] = 0$. Solve for $\dot{\lambda}$ and back

substitute to get solution for stress rate: $\dot{\underset{\approx}{\sigma}} = \dot{\underset{\approx}{\sigma}}^{\text{trial}} - \left(\dfrac{\underset{\approx}{B}:\dot{\underset{\approx}{\sigma}}^{\text{trial}}}{\underset{\approx}{B}:\underset{\approx}{A}}\right)\underset{\approx}{A}$.

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Slightly rearrange solution to final form:

$$\dot{\underset{\approx}{\sigma}} = P(\dot{\underset{\approx}{\sigma}}^{\text{trial}}) \qquad where \qquad P(\underset{\approx}{X}) = \underset{\approx}{X} - \frac{\underset{\approx}{A}(\underset{\approx}{B}:\underset{\approx}{X})}{\underset{\approx}{A}:\underset{\approx}{B}}$$
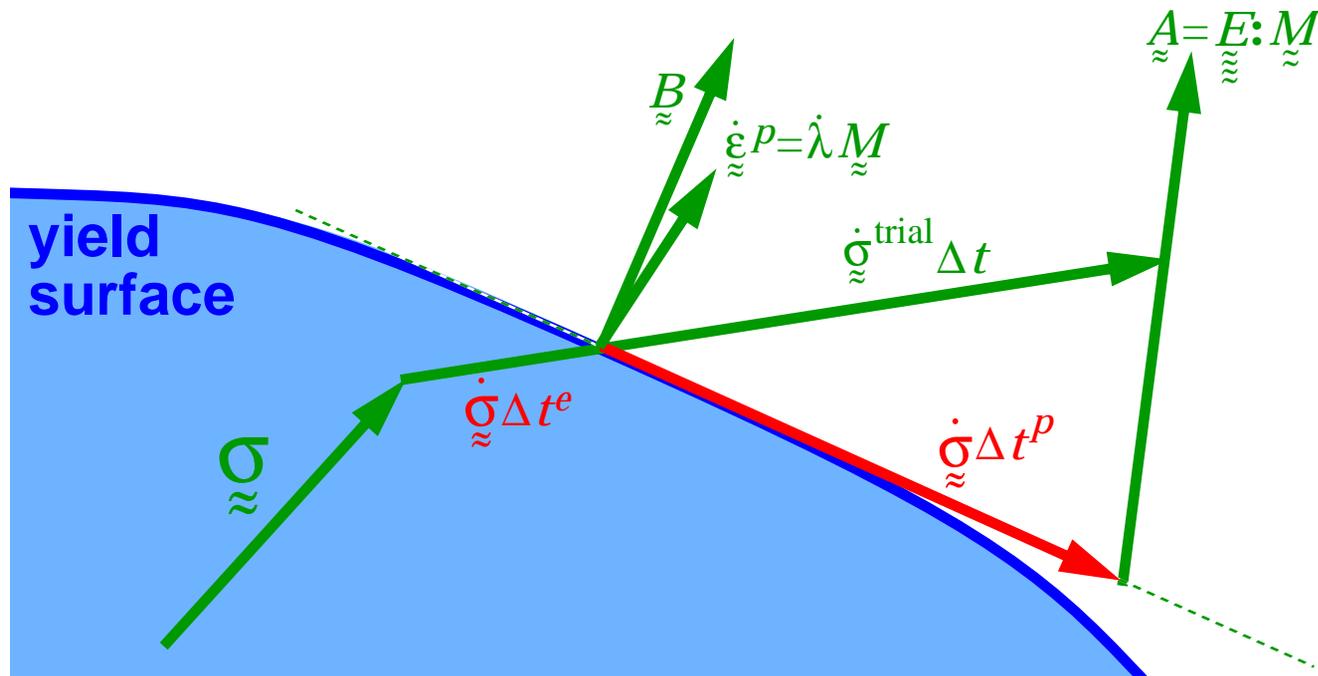
Numerical solution: $\underset{\approx}{\sigma} = \underset{\approx}{\sigma}^{\text{trial}} + \beta \underset{\approx}{A}$. Find β by $f(\underset{\approx}{\sigma}^{\text{trial}} + \beta \underset{\approx}{A}) = 0$.

Discussion

The return direction is...

- coaxial with A $(= \mathbf{E}:\mathbf{M})$.

- not generally normal to the yield surface.
- not generally aligned with the plastic strain rate.
- not dictated by physical considerations such as positive dissipation, yield surface convexity, or plastic stability. (Such concerns dictate appropriate values for "known" quantities.)
- "radial" if and only if the material is plastically incompressible.
- An algorithm that returns normal to the yield surface (i.e., $A = \alpha \mathbf{B}$) is *implicitly* using a plastic strain rate direction $\mathbf{M} = \alpha \mathbf{E}^{-1}:\mathbf{B}$.

The above analysis can be generalized (see web document) to include hardening/softening. Projection of the trial stress back to the *current* yield surface remains valid even though the stress rate is no longer a projection of the trial stress rate.

-72 of 93-

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Equivalent plastic strain

Many constitutive models use yield surface evolution laws that depend on the so-called "equivalent plastic strain," which is defined

$$\gamma_p \equiv \int \sqrt{\frac{2}{3}\dot{\underset{\approx}{\varepsilon}}{}^{p\prime}:\dot{\underset{\approx}{\varepsilon}}{}^{p\prime}}\,dt = \sqrt{\frac{2}{3}}\int \left\|\dot{\underset{\approx}{\varepsilon}}{}^{p\prime}\right\|dt$$

The best method uses the definition directly:

$$\Delta\gamma_p \equiv \sqrt{\frac{2}{3}}\left\|\dot{\underset{\approx}{\varepsilon}}{}' - \dot{\underset{\approx}{\varepsilon}}{}^{e\prime}\right\|\Delta t, \text{ or, for isotropic, } \Delta\gamma_p \equiv \sqrt{\frac{2}{3}}\left\|\dot{\underset{\approx}{\varepsilon}}{}' - \frac{\dot{\underset{\approx}{S}}}{2G}\right\|\Delta t.$$
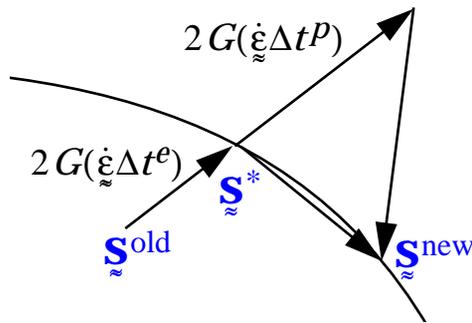
For a finite time step Δt,

$2G(\dot{\underset{\approx}{\varepsilon}}\Delta t^p)$

$2G(\dot{\underset{\approx}{\varepsilon}}\Delta t^e)$

$\underset{\approx}{S}^*$

$\underset{\approx}{S}^{old}$

$\underset{\approx}{S}^{new}$

$$\gamma_p^{new} \equiv \gamma_p^{old} + \sqrt{\frac{2}{3}}\left\|\dot{\underset{\approx}{\varepsilon}}{}'\Delta t - \frac{\underset{\approx}{S}^{new} - \underset{\approx}{S}^{old}}{2G}\right\|$$

(...better suited for partially plastic intervals.)

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Supplemental topic: invariant yield functions

Tresca: Stress state is below yield *if and only if*

Good

$$f(\underset{\sim}{\sigma}) = \frac{1}{2}\max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) - k < 0 \qquad (1)$$

Some authors (e.g. Fung, 1965, Lubliner 1990) **wrongly** claim that an acceptable *alternative* Tresca yield function is

BAD

$$F(\underset{\sim}{\sigma}) = [(\sigma_1 - \sigma_2)^2 - 4k^2][(\sigma_2 - \sigma_3)^2 - 4k^2][(\sigma_3 - \sigma_1)^2 - 4k^2]. \qquad (2)$$

This is intoxicating because it can be written with invariants as

WORSE

$$F(\underset{\sim}{\sigma}) = 4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 \qquad (3)$$

FATAL FLAW: If stress is below yield, then $F(\underset{\sim}{\sigma}) \leq 0$, but converse is false! A return algorithm using F might wrongly think a plastic trial stress is below yield. For example, $\sigma_1 = \sigma_2 = 3k$ and $\sigma_3 = 0$ is correctly identified to be above yield by $f(\underset{\sim}{\sigma})$, but not by $F(\underset{\sim}{\sigma})$.

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Under the assumption of plane stress where $\sigma_3 = 0$, regions where $F(\underset{\sim}{\sigma}) > 0$ are shown in black. *A valid yield function should be black everywhere outside the yellow Tresca hexagon*. The invariant $F(\underset{\sim}{\sigma})$ is invalid!

$\sigma_2/2k$

$\sigma_1/2k$

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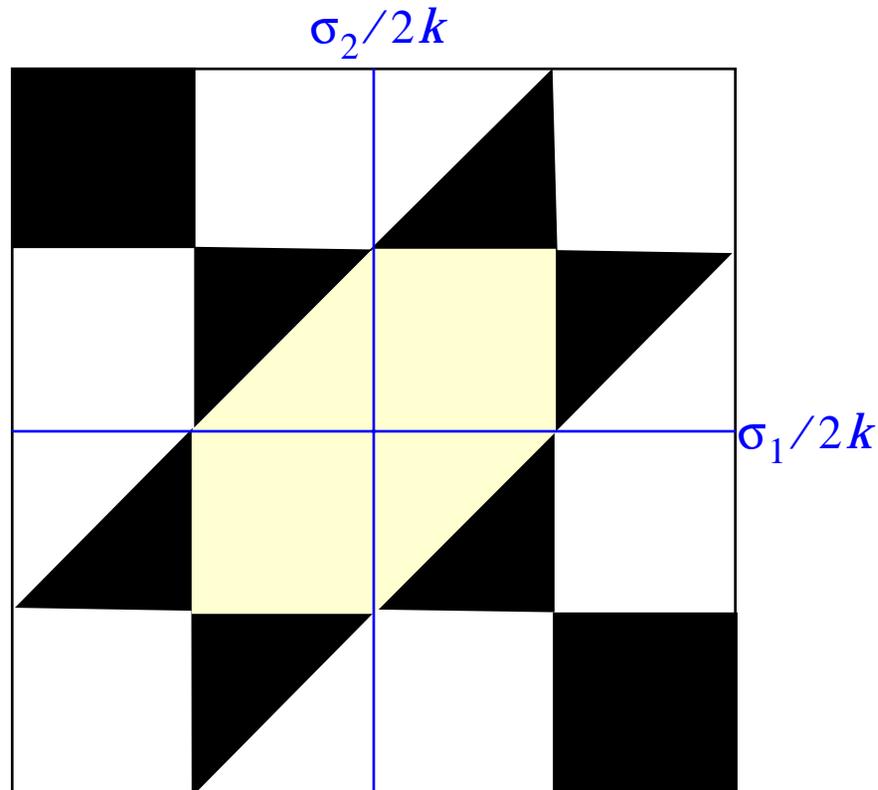
Supplemental topic: 9D vector basis

Recall that tensors are 9D vectors, so we may define a 9×1 component array for them: $T_1^o,\ T_2^o,\ T_3^o,\ T_4^o,\ T_5^o,\ T_6^o,\ T_7^o,\ T_8^o,\ T_9^o =$

$\{T_{11},\ T_{21},\ T_{31},\ T_{12},\ T_{22},\ T_{32},\ T_{13},\ T_{23},\ T_{33}\}$,

3D vector basis expansion	**9D tensor expansion**

3D vector basis expansion

$$\underset{\sim}{v} = v_1 \underset{\sim}{e}_1 + v_2 \underset{\sim}{e}_2 + v_3 \underset{\sim}{e}_3$$

9D tensor expansion

$$\underset{\approx}{T} = T_{11}\underset{\sim}{e}_1\underset{\sim}{e}_1 + T_{12}\underset{\sim}{e}_1\underset{\sim}{e}_2 + T_{13}\underset{\sim}{e}_1\underset{\sim}{e}_3$$
$$+ \ T_{21}\underset{\sim}{e}_2\underset{\sim}{e}_1 + T_{22}\underset{\sim}{e}_2\underset{\sim}{e}_2 + T_{23}\underset{\sim}{e}_2\underset{\sim}{e}_3$$
$$+ \ T_{31}\underset{\sim}{e}_3\underset{\sim}{e}_1 + T_{32}\underset{\sim}{e}_3\underset{\sim}{e}_2 + T_{33}\underset{\sim}{e}_3\underset{\sim}{e}_3$$

Summation form

$$\underset{\sim}{v} = \sum_{k=1}^{3} v_k \underset{\sim}{e}_k$$

Summation form

$$\underset{\approx}{T} = \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\underset{\sim}{e}_i\underset{\sim}{e}_j = \sum_{K=1}^{9} T_K^o \underset{\approx}{\xi}_K^o$$

where $T_1^o = T_{11}$, $\underset{\approx}{\xi}_1^o = \underset{\sim}{e}_1\underset{\sim}{e}_1$, $T_2^o = T_{21}$, $\underset{\approx}{\xi}_2^o = \underset{\sim}{e}_2\underset{\sim}{e}_1$,   etc.

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Subspace of symmetric tensors

Suppose that a physical problem involves a plane even if there are some non-planar aspects of the motion (e.g., oblique impact of a projectile onto a slab of armor). For solving the problem, any sensible engineer would line up a basis with the plane: all base vectors are either in the plane or normal to the plane.

The set of all symmetric tensors forms a *subspace*, which is analogous to a plane. The "normal" to the plane is the set of all skew-symmetric tensors. If you add two vectors in a plane, the result is also in the plane. Analogously, if you form any linear combination of symmetric tensors, the result is also symmetric.

Yield functions are defined for stress, which is symmetric. Our constitutive modelling problems intimately involve symmetric tensors, so it makes sense to use a basis for tensor space such that all base tensors are either purely symmetric or purely skew-symmetric.

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Voigt components: $\{T\}^v = \{T_{11},\, T_{22},\, T_{33},\, T_{23},\, T_{31},\, T_{12}\}$

Then $\underset{\approx}{R} \!:\! \underset{\approx}{S}$ equals $R_1^v S_1^v + R_2^v S_2^v + R_3^v S_3^v + \quad \mathbf{2}(R_4^v S_4^v + R_5^v S_4^v + R_6^v S_6^v)$

Note the ungainly factor of 2 needed because the off diagonal components contribute *twice* in the expression $\underset{\approx}{R} \!:\! \underset{\approx}{S} = R_{ij} S_{ij}$.

Mandel components: $\{T\}^m = \{T_{11},\, T_{22},\, T_{33},\, \sqrt{2}\,T_{23},\, \sqrt{2}\,T_{31},\, \sqrt{2}\,T_{12}\}$

Then $\underset{\approx}{R} \!:\! \underset{\approx}{S}$ equals $R_1^m S_1^m + R_2^m S_2^m + R_3^m S_3^m + R_4^m S_4^m + R_5^m S_4^m + R_6^m S_6^m$

Ah! much more intuitive! The Mandel approach incorporates the factor of 2 inside the definition of the components.

Q: Is the Mandel convention just a "trick" likely to bite us some day?

A: NO! Voigt components are the dangerous choice — they are referenced to an irregular basis for symmetric tensors. Mandel components are referenced to the same — *but normalized* — basis!

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The basis expansion of *any* tensor may be written

$$\underset{\approx}{T} = T_{11}\mathbf{e}_1\mathbf{e}_1 + T_{12}\mathbf{e}_1\mathbf{e}_2 + T_{13}\mathbf{e}_1\mathbf{e}_3 + T_{21}\mathbf{e}_2\mathbf{e}_1 + T_{22}\mathbf{e}_2\mathbf{e}_2 + T_{23}\mathbf{e}_2\mathbf{e}_3 + T_{31}\mathbf{e}_3\mathbf{e}_1 + T_{32}\mathbf{e}_3\mathbf{e}_2 + T_{33}\mathbf{e}_3\mathbf{e}_3$$

$$= T_{(11)}\mathbf{e}_1\mathbf{e}_1 + T_{(22)}\mathbf{e}_2\mathbf{e}_2 + T_{(33)}\mathbf{e}_3\mathbf{e}_3$$
$$+ T_{(23)}(\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2) + T_{(31)}(\mathbf{e}_3\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_3) + T_{(12)}(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1)$$
$$+ T_{[32]}(\mathbf{e}_3\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_3) + T_{[13]}(\mathbf{e}_1\mathbf{e}_3 - \mathbf{e}_3\mathbf{e}_1) + T_{[21]}(\mathbf{e}_2\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2)$$

where

$$T_{(ij)} \equiv \frac{1}{2}(T_{ij} + T_{ji}) \ and \ T_{[ij]} \equiv \frac{1}{2}(T_{ij} - T_{ji})$$

If the tensor is symmetric, the last three terms are all zero. If the tensor is skew-symmetric, then the first six terms are all zero and the last three terms are the components of the axial vector.

$$\underset{\approx}{T} = T_{(11)}\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_1 + T_{(22)}\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_2 + T_{(33)}\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_3$$

$$+ T_{(23)}(\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3 + \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2) + T_{(31)}(\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3) + T_{(12)}(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2 + \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_1)$$

$$+ T_{[32]}(\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2 - \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3) + T_{[13]}(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3 - \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1) + T_{[21]}(\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_1 - \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2)$$

Traditional Voigt:

$$T_1^v = T_{(11)}, \quad T_2^v = T_{(22)}, \quad T_3^v = T_{(33)}, \quad T_4^v = T_{(23)}, \qquad T_5^v = T_{(31)}, \ ...$$

$$\underset{\approx}{\xi}_1^v = \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_1, \quad \underset{\approx}{\xi}_2^v = \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_2, \quad \underset{\approx}{\xi}_3^v = \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_3, \quad \underset{\approx}{\xi}_4^v = (\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3 + \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2), \quad \underset{\approx}{\xi}_5^v = (\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3), \ ...$$

Then $\displaystyle \underset{\approx}{T} = \sum_{K=1}^{9} T_K^v \underset{\approx}{\xi}_K^v$.

For symmetric, $T_{(ij)} = T_{ij}$ and $T_{[ij]} = 0$, so the sum may be truncated at six terms.

MAJOR PROBLEM: Voigt basis is not normalized!

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Voigt basis is not normalized

Consider the inner product:

$$\underset{\approx}{R} : \underset{\approx}{S} = \left(\sum_{K=1}^{9} R_K^v \underset{\approx}{\xi}_K^v \right) : \left(\sum_{J=1}^{9} S_J^v \underset{\approx}{\xi}_J^v \right) = \sum_{K=1}^{9} \sum_{J=1}^{9} R_K^v S_J^v (\underset{\approx}{\xi}_K^v : \underset{\approx}{\xi}_J^v)$$

The Voigt basis is orthogonal:

$$\underset{\approx}{\xi}_K^v : \underset{\approx}{\xi}_J^v = 0 \text{ if } K \neq J.$$

The first three Voigt base tensors are normalized:

$\underset{\approx}{\xi}_1^v : \underset{\approx}{\xi}_1^v = 1$, $\underset{\approx}{\xi}_2^v : \underset{\approx}{\xi}_2^v = 1$, and $\underset{\approx}{\xi}_3^v : \underset{\approx}{\xi}_3^v = 1$, but the remaining base tensors are not normalized. They all have a magnitude of $\sqrt{2}$. Thus

$$\underset{\approx}{R} : \underset{\approx}{S} = \sum_{K=1}^{9} R_K^v S_K^v \left\| \underset{\approx}{\xi}_K \right\|^2 = R_1^v S_1^v + R_2^v S_2^v + R_3^v S_3^v + 2(R_4^v S_4^v) + 2(R_5^v S_5^v) + \dots$$

Obvious thing to do ... normalize the basis.

Mandel basis: $\underset{\approx}{\xi}_K^m \equiv \dfrac{\underset{\approx}{\xi}_K^v}{\left\|\underset{\approx}{\xi}_K^v\right\|}$.

$T_1^m = T_{(11)},\ T_2^m = T_{(22)},\ T_3^m = T_{(33)},\ T_4^m = \sqrt{2}\, T_{(23)}, \qquad T_5^m = \sqrt{2}\, T_{(31)},\ \ldots$

$\underset{\approx}{\xi}_1^m = \underset{\sim}{\mathbf{e}}_1 \underset{\sim}{\mathbf{e}}_1,\ \ \underset{\approx}{\xi}_2^m = \underset{\sim}{\mathbf{e}}_2 \underset{\sim}{\mathbf{e}}_2,\ \ \underset{\approx}{\xi}_3^m = \underset{\sim}{\mathbf{e}}_3 \underset{\sim}{\mathbf{e}}_3,\ \ \underset{\approx}{\xi}_4^m = \dfrac{(\underset{\sim}{\mathbf{e}}_2 \underset{\sim}{\mathbf{e}}_3 + \underset{\sim}{\mathbf{e}}_3 \underset{\sim}{\mathbf{e}}_2)}{\sqrt{2}},\ \ \underset{\approx}{\xi}_5^m = \dfrac{(\underset{\sim}{\mathbf{e}}_3 \underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_1 \underset{\sim}{\mathbf{e}}_3)}{\sqrt{2}},\ \ \ldots$

Then $\quad \underset{\approx}{T} = \displaystyle\sum_{K=1}^{9} T_K^m \underset{\approx}{\xi}_K^m, \qquad \underset{\approx}{\xi}_K^m : \underset{\approx}{\xi}_J^m = \delta_{KJ}, \qquad$ and $\qquad \underset{\approx}{R} : \underset{\approx}{S} = \displaystyle\sum_{K=1}^{9} R_K^m S_K^m$

With this orthonormal Mandel basis, the tensor inner product takes a form that is a direct analog of the ordinary 3D vector inner product formula that applies when the basis is orthonormal.

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Mandel basis for symmetric tensors

Dropping "m" identifier, the Mandel basis for 9D full tensor space is

$$\underset{\approx}{\xi}_1 = \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_1, \qquad \underset{\approx}{\xi}_2 = \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_2, \qquad \underset{\approx}{\xi}_3 = \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_3$$

$$\underset{\approx}{\xi}_4 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3 + \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2), \qquad \underset{\approx}{\xi}_5 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1 + \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3), \qquad \underset{\approx}{\xi}_6 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2 + \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_1)$$

$$\underset{\approx}{\xi}_7 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_2 - \underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_3), \qquad \underset{\approx}{\xi}_8 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_3 - \underset{\sim}{\mathbf{e}}_3\underset{\sim}{\mathbf{e}}_1), \qquad \underset{\approx}{\xi}_9 = \frac{1}{\sqrt{2}}(\underset{\sim}{\mathbf{e}}_2\underset{\sim}{\mathbf{e}}_1 - \underset{\sim}{\mathbf{e}}_1\underset{\sim}{\mathbf{e}}_2)$$

The basis is orthogonal because $\underset{\approx}{\xi}_K : \underset{\approx}{\xi}_J = 0$ if $K \neq J$. The basis is *normalized* (i.e., $\underset{\approx}{\xi}_K : \underset{\approx}{\xi}_J = \delta_{KJ}$) because of the factors of $\sqrt{2}$.

Just as an ordinary vector has components $v_k = \underset{\sim}{\mathbf{v}} \bullet \underset{\sim}{\mathbf{e}}_k$, the Mandel components of a tensor $\underset{\approx}{T}$ are $T_K = \underset{\approx}{T} : \underset{\approx}{\xi}_K$.

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Stress: $\underset{\approx}{\sigma}$

Mean stress: $p = \dfrac{1}{3}\mathrm{tr}\underset{\approx}{\sigma} = \dfrac{1}{3}\underset{\approx}{\mathbf{I}}:\underset{\approx}{\sigma}$ (positive in tension)

Stress deviator: $\underset{\approx}{\mathbf{S}} = \underset{\approx}{\sigma} - p\underset{\approx}{\mathbf{I}}$

Magnitude of the stress deviator: $\tau \equiv \sqrt{\underset{\approx}{\mathbf{S}}:\underset{\approx}{\mathbf{S}}}$

Unit tensor in the direction of $\underset{\approx}{\mathbf{S}}$: $\hat{\underset{\approx}{\mathbf{S}}} \equiv \dfrac{\underset{\approx}{\mathbf{S}}}{\|\underset{\approx}{\mathbf{S}}\|} = \dfrac{\underset{\approx}{\mathbf{S}}}{\sqrt{\underset{\approx}{\mathbf{S}}:\underset{\approx}{\mathbf{S}}}} = \dfrac{\underset{\approx}{\mathbf{S}}}{\tau}$

Then $\underset{\approx}{\sigma} = \tau\hat{\underset{\approx}{\mathbf{S}}} + p\underset{\approx}{\mathbf{I}}$.

We now show that non-intuitive factors appear because the identity $\underset{\approx}{\mathbf{I}}$ is not a unit tensor. Specifically, $\|\underset{\approx}{\mathbf{I}}\| = \sqrt{\underset{\approx}{\mathbf{I}}:\underset{\approx}{\mathbf{I}}} = \sqrt{3}$.

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Motivational example

A popular simplified yield criterion assumes that the yield function depends only on τ and p. $F(\underset{\approx}{\sigma}) = f(\tau, p)$. The yield surface defined by $F(\underset{\approx}{\sigma}) = 0$ is a hyper-cylinder in stress space — it is a surface of revolution about the isotropic axis.

Gradient of yield: $\underset{\approx}{\mathbf{B}} = \dfrac{dF}{d\underset{\approx}{\sigma}} = \dfrac{\partial f}{\partial \tau}\left(\dfrac{d\tau}{d\underset{\approx}{\sigma}}\right) + \dfrac{\partial f}{\partial p}\left(\dfrac{dp}{d\underset{\approx}{\sigma}}\right) = \dfrac{\partial f}{\partial \tau}(\underset{\approx}{\hat{\mathbf{S}}}) + \dfrac{\partial f}{\partial p}\left(\dfrac{1}{3}\underset{\approx}{\mathbf{I}}\right)$

Let $\underset{\approx}{\sigma}^t = \tau^t\underset{\approx}{\hat{\mathbf{S}}} + p^t\underset{\approx}{\mathbf{I}}$ denote a trial (t) elastic stress.

Let $\underset{\approx}{\sigma}^n = \tau^n\underset{\approx}{\hat{\mathbf{S}}} + p^n\underset{\approx}{\mathbf{I}}$ denote the new (n) updated stress on the yield surface obtained by returning to the *nearest point* on the yield surface in stress space.   (We now know this doesn't necessarily mean that the plastic strain rate is normal to the yield surface — we use a normal return direction to illustrate a different point here. A normal return direction implies a direction of plastic strain rate parallel to $\underset{\approx}{\mathbf{E}}^{-1}:\underset{\approx}{\mathbf{B}}$).

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Suppose we wish to return *nearest point* on yield surface. Then $\underset{\approx}{\sigma}^t - \underset{\approx}{\sigma}^n$ is normal to the yield surface. There's a scalar β such that

$$\underset{\approx}{\sigma}^t - \underset{\approx}{\sigma}^n = \beta\underset{\approx}{\mathbf{B}}, \quad \text{or} \quad (\tau^t - \tau^n)\underset{\approx}{\hat{\mathbf{S}}} + (p^t - p^n)\underset{\approx}{\mathbf{I}} = \beta\left(\frac{\partial f}{\partial \tau}(\underset{\approx}{\hat{\mathbf{S}}}) + \frac{\partial f}{\partial p}\left(\frac{1}{3}\underset{\approx}{\mathbf{I}}\right)\right)$$

Therefore $\dfrac{\tau^t - \tau^n}{p^t - p^n} = 3\left(\dfrac{\partial f/\partial \tau}{\partial f/\partial p}\right)$.

Thus, to project normal to the yield surface *in stress space*, you must project using a slope *3 times steeper* than the normal in τ vs. p space. This counterintuitive behavior arises because τ and p are not isomorphic to stress space. The base tensors $\underset{\approx}{\hat{\mathbf{S}}}$ and $\underset{\approx}{\mathbf{I}}$, while orthogonal, are not normalized. We should instead use $\underset{\approx}{\hat{\mathbf{I}}} \equiv \underset{\approx}{\mathbf{I}}/\left\|\underset{\approx}{\mathbf{I}}\right\| = \underset{\approx}{\mathbf{I}}/\sqrt{3}$ with an appropriately modified measure of mean stress. Namely, $\hat{p} = \sqrt{3}\,p$.

trial stress (p^t, τ^t)

wrong answer

correct answer (p^n, τ^n)

GOAL: RETURN TO NEAREST POINT ON THE YIELD SURFACE *in stress space!*

That means project *obliquely* in this τ v.s. p space.

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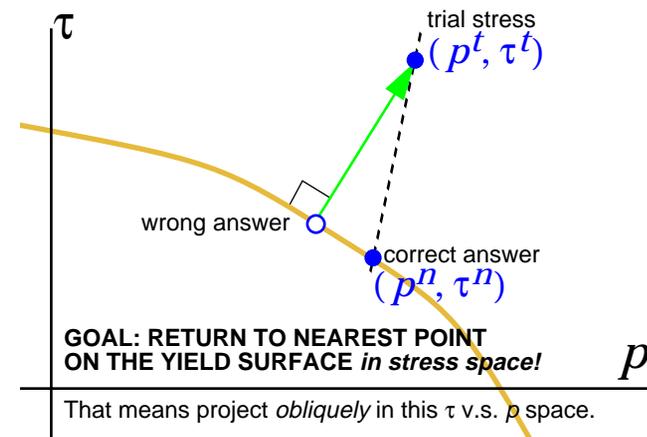
Rendulic plane

The Rendulic plane plots a "shear stress" versus a "mean stress."

Engineer's choice

"shear stress:" $\tau = \sqrt{\underset{\approx}{\mathbf{S}}:\underset{\approx}{\mathbf{S}}}$, and

"mean stress:" $p = \dfrac{1}{3}\mathrm{tr}\underset{\approx}{\sigma}$. Then $\underset{\approx}{\sigma} = \underset{\approx}{\mathbf{S}} + p\underset{\approx}{I}$.

Problem: This τ vs. p space isn't isomorphic to stress space. For example, $\underset{\approx}{\sigma}:\underset{\approx}{\sigma} \neq \tau^2 + p^2$. Importantly, the normal to the yield surface in τ vs. p space is *not* normal to the yield surface in stress space.

View in 3-D physical space

$\underset{\sim}{\mathbf{x}} = r\hat{\underset{\sim}{\mathbf{e}}}_r + z\hat{\underset{\sim}{\mathbf{e}}}_z$

Cylindrical z vs. r plane.

$z\hat{\underset{\sim}{\mathbf{e}}}_z$

$\underset{\sim}{X}$

$\hat{\underset{\sim}{\mathbf{e}}}_z$

$\hat{\underset{\sim}{\mathbf{e}}}_r$ $r\hat{\underset{\sim}{\mathbf{e}}}_r$

isomorphic

engineering

τ

p or $\hat{p}$
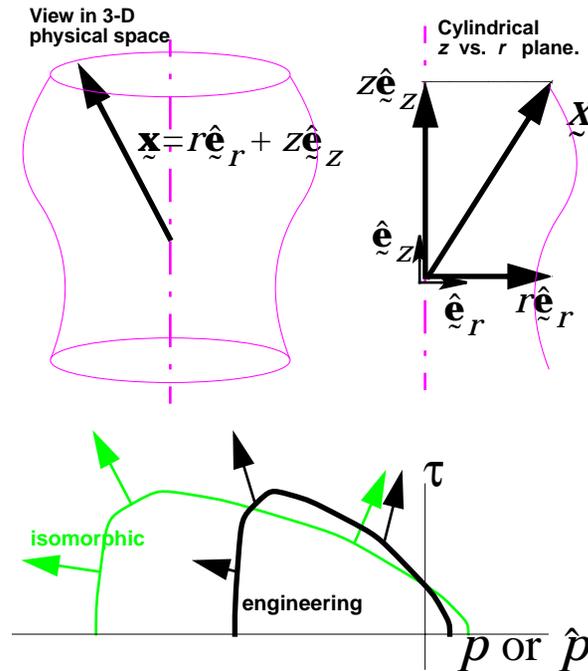
Mathematician's (isomorphic) choice: "shear stress"

$\tau = \sqrt{\underset{\approx}{\mathbf{S}}:\underset{\approx}{\mathbf{S}}} = \underset{\approx}{\sigma}:\hat{\underset{\approx}{\mathbf{S}}}$ and "mean stress" $\hat{p} = \dfrac{1}{\sqrt{3}}\mathrm{tr}\underset{\approx}{\sigma} = \underset{\approx}{\sigma}:\hat{\underset{\approx}{\mathbf{I}}} = \sqrt{3}\,p$. Then

$\underset{\approx}{\sigma} = \tau\hat{\underset{\approx}{\mathbf{S}}} + \hat{p}\hat{\underset{\approx}{I}}$. The normalized $\hat{\underset{\approx}{I}}$ is like the $\hat{\underset{\sim}{\mathbf{e}}}_z$ cylindrical base vector.

Supplemental Topic:
Anisotropic yield surfaces

For elastically anisotropic material, a very common "first-cut" best guess at the plastic yield surface is a Tsai-Wu ellipsoid of the form

$$f(\underset{\sim}{\sigma}) = (\underset{\sim}{\sigma} - \underset{\sim}{\sigma}*):\underset{\approx}{L}:(\underset{\sim}{\sigma} - \underset{\sim}{\sigma}*) - 1 \, ,$$ *(contrary to Walker's recent claims, this form is perfectly capable of modelling even highly anisotropic media.)*

where $\underset{\approx}{L}$ shares the same anisotropy with the stiffness $\underset{\approx}{E}$.

Elastic constants may be *nondestructively* measured, but the yield L_{ijkl} parameters are more difficult since a fresh sample must be used to measure each component. Thus, data are often lacking.

Proposal: Face with a dearth of data, assume that $\underset{\approx}{E}$ and $\underset{\approx}{L}$ have the same *eigenprojectors*, a term which we now define...

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What are eigenprojectors?

To illustrate, consider simpler 3D space. Here's a sample tensor

$$[\underset{\approx}{\mathbf{A}}] = \begin{bmatrix} 17 & -2 & -2 \\ -2 & 14 & -4 \\ -2 & -4 & 14 \end{bmatrix}, \textit{ which has eigenpairs } \begin{array}{ll} \lambda_1 = 9 & \mathbf{v}_1 = \frac{1}{3}\{1, 2, 2\} \\ \lambda_2 = 18 & \mathbf{v}_2 = \frac{1}{\sqrt{5}}\{-2, 0, 1\} \\ \lambda_3 = 18 & \mathbf{v}_3 = \frac{1}{3\sqrt{5}}\{-2, 5, -4\} \end{array}$$

In spectral form, $\underset{\approx}{\mathbf{A}} = \lambda_1 \mathbf{v}_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 \mathbf{v}_3$

$$= 9 \underbrace{\mathbf{v}_1 \mathbf{v}_1}_{\underset{\approx}{P}_1} + 18 \underbrace{(\mathbf{v}_2 \mathbf{v}_2 + \mathbf{v}_3 \mathbf{v}_3)}_{\underset{\approx}{P}_2} \qquad \longleftarrow \text{ unique!}$$

With respect to the principal basis,

$$\underset{\approx}{\mathbf{A}} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}, \qquad \underset{\approx}{P}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \quad \underset{\approx}{P}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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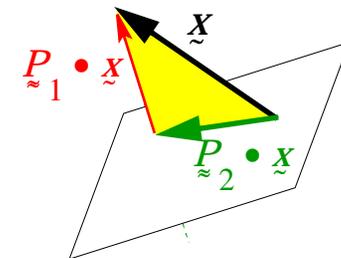
We seek tensors $\underset{\approx}{Y}$ and scalars λ such that $\underset{\approx}{\mathbf{E}} : \underset{\approx}{Y} = \lambda \underset{\approx}{Y}$. The major

and minor symmetries of $\underset{\approx}{\mathbf{E}}$ allow this to be written as an ordinary

6×6 matrix eigenproblem:

$$
\begin{bmatrix}
E_{1111} & E_{1122} & E_{1133} & \sqrt{2}E_{1123} & \sqrt{2}E_{1131} & \sqrt{2}E_{1112} \\
E_{2211} & E_{2222} & E_{2233} & \sqrt{2}E_{2223} & \sqrt{2}E_{2231} & \sqrt{2}E_{2212} \\
E_{3311} & E_{3322} & E_{3333} & \sqrt{2}E_{3323} & \sqrt{2}E_{3331} & \sqrt{2}E_{3312} \\
\sqrt{2}E_{2311} & \sqrt{2}E_{2322} & \sqrt{2}E_{2333} & 2E_{2323} & 2E_{2331} & 2E_{2312} \\
\sqrt{2}E_{3111} & \sqrt{2}E_{3122} & \sqrt{2}E_{3133} & 2E_{3123} & 2E_{3131} & 2E_{3112} \\
\sqrt{2}E_{1211} & \sqrt{2}E_{1222} & \sqrt{2}E_{1233} & 2E_{1223} & 2E_{1231} & 2E_{1212}
\end{bmatrix}
\begin{bmatrix}
Y_{11} \\ Y_{22} \\ Y_{33} \\ \sqrt{2}Y_{23} \\ \sqrt{2}Y_{31} \\ \sqrt{2}Y_{23}
\end{bmatrix}
= \lambda
\begin{bmatrix}
Y_{11} \\ Y_{22} \\ Y_{33} \\ \sqrt{2}Y_{23} \\ \sqrt{2}Y_{31} \\ \sqrt{2}Y_{23}
\end{bmatrix}
$$

An eigensolver will output a set of six orthonormal 6-dimensional eigenvectors. Each of these correspond to symmetric eigentensors.

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If λ has multiplicity of 1, then $P_{ijkl} = Y_{ij}Y_{kl}$ is the corresponding eigenprojector. When it operates on any tensor, the result is the part of that tensor in the direction of Y_{ij}.

EXAMPLE: For isotropy, 3K is an eigenvalue of multiplicity 1. The *normalized* eigentensor is $\underset{\approx}{I}/\sqrt{3}$. The projector is $\frac{1}{3}\delta_{ij}\delta_{kl}$, which merely returns the isotropic part of any tensor it operates on.

If λ has multiplicity of 2, then the eigentensors $\underset{\approx}{Y}^{(1)}$ and $\underset{\approx}{Y}^{(2)}$ are not unique. Instead, the eigenprojector, $P_{ijkl} = Y_{ij}^{(1)}Y_{kl}^{(1)} + Y_{ij}^{(2)}Y_{kl}^{(2)}$ is unique. When it operates on an arbitrary tensor, the result is the part of the tensor in the subspace. Higher multiplicities are similar.

EXAMPLE: For isotropy, 2G is an eigenvalue of multiplicity 5. The eigenprojector (constructed by summing dyads of the five *orthonormalized* eigenprojectors) returns the deviator of any tensor it operates on. Thus, ANY DEVIATORIC TENSOR is an eigentensor for isotropy.

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Recall $f(\underset{\approx}{\sigma}) = (\underset{\approx}{\sigma} - \underset{\approx}{\sigma}*):\underset{\approx}{\mathbf{L}}:(\underset{\approx}{\sigma} - \underset{\approx}{\sigma}*) - 1$. If the material is transverse, the Mandel eigenproblem is of the form

$$
\begin{bmatrix}
E_0 & E_2 & E_3 & 0 & 0 & 0 \\
E_2 & E_0 & E_3 & 0 & 0 & 0 \\
E_3 & E_3 & E_1 & 0 & 0 & 0 \\
0 & 0 & 0 & E_4 & 0 & 0 \\
0 & 0 & 0 & 0 & E_5 & 0 \\
0 & 0 & 0 & 0 & 0 & E_5
\end{bmatrix}
\begin{bmatrix}
Y_{11} \\
Y_{22} \\
Y_{33} \\
\sqrt{2}\,Y_{23} \\
\sqrt{2}\,Y_{31} \\
\sqrt{2}\,Y_{23}
\end{bmatrix}
= \lambda
\begin{bmatrix}
Y_{11} \\
Y_{22} \\
Y_{33} \\
\sqrt{2}\,Y_{23} \\
\sqrt{2}\,Y_{31} \\
\sqrt{2}\,Y_{23}
\end{bmatrix},
\quad \text{where}
$$

$E_1 = E_{3333}, \quad E_2 = E_{1122}, \quad E_3 = E_{1133},$

$E_4 = 2E_{2323}, \quad E_5 = 2E_{1212}, \quad E_o = E_2 + E_5$

There are five independent stiffnesses, but only four independent eigenvalues (and therefore only four independent eigenprojectors). Forcing $\underset{\approx}{\mathbf{L}}$ to have the same eigenprojectors gives a formula for the elusive L_{1133} value that couples lateral and axial response.

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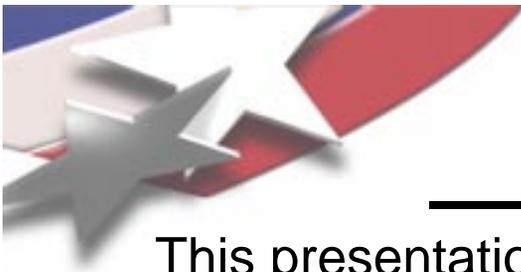
Conclusions

This presentation covered many applications that illustrate the usefulness of regarding tensors as higher-dimensional vectors.

Key points were

- For radial and oblique return models, the stress may be returned to the yield surface via a projection operation that is analogous to projecting a simple vector onto a plane.
- Symmetric tensors are analogous to planes. The Mandel convention for symmetric tensor components correspond to an ortho*normal* basis for symmetric tensors.
- The invariant form of the Tresca yield criterion is invalid because negative values of that "yield function" do not necessarily correspond to stresses that are below yield.
- The isomorphic stress measures are a more accurate representation of stress space that is analogous to viewing the stress "vector" in the "plane" formed by the isotropic tensor and the stress itself.
- Anisotropic yield may be coupled to elastic isotropy via the elastic eigenprojectors.

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