J. Mech. Phys. Solids Vol. 41, No. 2, pp. 297–330, 1993.
Printed in Great Britain.

0022–5096/93 $6.00 + 0.00

INFLUENCE OF NON-CLASSICAL ELASTIC–PLASTIC CONSTITUTIVE FEATURES ON SHOCK WAVE EXISTENCE AND SPECTRAL SOLUTIONS

R. M. BRANNON and W. J. DRUGAN†

Department of Engineering Mechanics, University of Wisconsin—Madison, Madison, WI 53706, U.S.A.

(*Received* 5 *November* 1991; *in revised form* 30 *April* 1992)

ABSTRACT

THE INFLUENCE of non-classical elastic–plastic constitutive features on dynamically moving discontinuities in stress, strain and material velocity is investigated. Non-classical behavior here includes non-normality of the plastic strain increment to the yield surface, plastic compressibility, pressure sensitivity of yield and dependence of the elastic moduli on plastic strain. DRUGAN and SHEN's (1987) analysis of dynamically moving discontinuities with strain as well as stress jumps *in classical materials* is shown to be valid for a broad class of non-associative material models until deviation from normality exceeds a critical (non-infinitesimal) level. For these non-classical materials, an inequality that bounds the *magnitude* of the stress jump is derived, which is information not obtainable from a standard spectral analysis of a shock. For the special case of stress discontinuities with *continuous* strain or for quasi-static deformations, this inequality is shown to rule out jumps in specific projections of the stress tensor unless the non-normality is sufficiently large. These results invalidate a recent claim in the literature that an *infinitesimal* amount of non-normality permits moving surfaces of discontinuity in stress (with no strain jump) near the tip of a dynamically advancing crack tip. Using a very general plastic constitutive law that subsumes most non-classical (and classical) descriptions currently in use, a *complete* closed form solution is obtained for the plastic wave speeds and eigenvectors. A novel feature of the analysis is the clarity and completeness of the solutions. If the elastic part of the response is isotropic, one plastic wave speed equals the elastic shear wave speed, while the other two possible wave speeds depend in general on the stress and plastic strain within the shock transition layer. Concise necessary and sufficient conditions for real eigenvalues and for vanishing eigenvalues are derived. The real eigenvalues are classified by numerical sign and ordering relative to the elastic eigenvalues. The geometric multiplicity of plastic eigenvectors associated with elastic eigenvalues is shown to depend on the stress state within the shock transition layer. These solutions, several of which hold for arbitrary elastic anisotropy, are also applicable to acceleration waves and localization problems and to materials with dependence of the elastic moduli on plastic strain. Such elastic–plastic coupling is shown to imply a non-self-adjoint fourth order tangent stiffness tensor even if the plastic constitutive law is associative.

1. INTRODUCTION

IN THIS PAPER we analyze dynamically moving discontinuities in stress and/or strain (which we term "shocks") for elastic–plastic material models that possess one or more of the following non-conventional features.

- Non-normality of the plastic strain increment to the yield surface.
- Pressure sensitivity of yield.

† To whom correspondence should be addressed.

- Plastic compressibility (i.e. permanent volume change).
- Yield surface vertices.
- Elastic–plastic coupling: the elastic moduli change with plastic strain.

Quite recent research has shown that non-classical constitutive features such as these permit a realistic description of important physical phenomena such as plastic flow localization (HILL, 1952; HILL and HUTCHINSON, 1975; RICE, 1976; TVERGAARD *et al.*, 1981), general bifurcation (RANIECKI and BRUHNS, 1981), non-conventional plastic slip in single crystals (ASARO, 1979, 1983), porous metal instabilities (BECKER, 1987; OHNO and HUTCHINSON, 1984; TVERGAARD, 1990), the strength differential effect in metals (NEEDLEMAN and RICE, 1978; RICE, 1976), pressure sensitivity of yield and plastic compressibility caused by the presence of microvoids (BEREZIN, 1987; GURSON, 1977), and non-normality caused by internal friction (CHANDLER, 1985) or by microvoid nucleation (MRÓZ and RANIECKI, 1976; NEEDLEMAN and RICE, 1978; TVERGAARD, 1982).

Pressure insensitivity of yield is an excellent approximation for many materials such as undamaged metal alloys (BRIDGMAN, 1952; SPITZIG and RICHMOND, 1984). However, even these materials can become appreciably pressure sensitive whenever pores or microcracks develop. In this case, the use of a pressure sensitive yield criterion is imperative. The expansion, interaction and eventual coalescence of voids is a basic precursor to ductile fracture. Voids may be present as a result of the processing (e.g. sintered metals) or may develop due to, say, cracking or debonding of rigid inclusions in high triaxial regions such as the zone ahead of a crack tip. Microporous or microcracked media exhibit pressure sensitivity of yield, plastic compressibility and dependence of the elastic moduli on previous plastic straining. Furthermore, nucleation of voids or internal friction at microcracks can produce non-normality.

Discontinuities in stress and strain fields are intimately related to localization of deformation into thin shear bands. HILL (1952) was the first to recognize that this phenomenon can be treated as a constitutive instability. His excellent summary of discontinuity relations (HILL, 1961) set the stage for his general framework for the study of localization in connection with acceleration waves and stationary discontinuities (HILL, 1962). RICE (1976) also studied the sensitivity of localization predictions to deviations from classical plasticity—especially non-normality and vertices. HILL (1967) provided evidence that vertex effects can be significant in certain situations even for polycrystals. For example, although localized necking of thin homogeneous sheets under biaxial stretching is impossible when associative theories with smooth yield loci are used, STOREN and RICE (1975) predicted such localization by approximating the effect of a vertex as a non-normality of the plastic strain increment to a *smooth* yield surface.

The emphasis of this paper is to determine how deviations from classical plasticity might affect moving surfaces of discontinuity in stress, strain or velocity. Such information is essential for the rigorous analysis of any phenomenon that is suspected to involve such discontinuities. We will prove by contradiction that a large class of elastic–plastic constitutive models (including, but not limited to, most classical models) do not permit certain types of moving discontinuities in stress. Discontinuities appear in elastic–plastic problems such as dynamic crack propagation in ductile single

crystals (NIKOLIC and RICE, 1988), moving load problems and metal forming/processing. For these problems, we must know what conditions must be enforced across the discontinuity surfaces as well as the level of sophistication required of the constitutive model in order consistently to model the discontinuities at all. For numerical approaches, known or suspected jumps can be built into the solution, or other provisions can be made to ensure accurate jump treatment. If an analytical solution to a boundary value problem is sought, knowledge of whether discontinuities are possible will ensure that appropriate initial assumptions are made. Otherwise, it could occur that either no solutions are found, or, more insidiously, the solutions found are incomplete, or—perhaps worst of all—"solutions" possessing discontinuities that violate a governing principle are advanced.

Much work has already been done in the area of moving discontinuity analysis for elastic–plastic materials. Permitting yield surface vertices and flats, arbitrary elastic and initial plastic anisotropy, and a broad class of anisotropic hardening in an otherwise classical material class, DRUGAN and RICE (1984) showed that a moving discontinuity in *any* component of stress is impossible for quasi-static small deformations. DRUGAN (1986) streamlined this analysis and showed that the result also holds for elastic–plastic response where the elastic part is non-linear hyperelastic. DRUGAN and SHEN (1987) and SHEN and DRUGAN (1990) then proved that the same material *does* allow jumps across *dynamically* propagating surfaces, but only under specific, quite restrictive, conditions. Finally, DRUGAN and SHEN (1990) arrived at similar conclusions for finite plastic deformations, but showed that material anisotropy complicates the results in the large deformation case. Consistent results were obtained by LEIGHTON *et al.* (1987), who showed that yield surface convexity with plastic strain increment normality precludes the existence of discontinuous plastic fields near the tip of a dynamically growing crack in the special case of steady-state plane-strain small deformations of an incompressible, isotropic, elastic–ideally plastic material.

All previous work of Drugan, Leighton and co-workers involved the key idea of enforcing an integrated form of the Maximum Plastic Work Inequality (MPWI) throughout passage of the shock. However, because the MPWI embodies both *convexity* of the yield surface and *normality* of the plastic strain increment to the yield surface, it cannot be used for non-associative (non-normal) materials. *In this paper, we release the requirement of normality and require only that the yield surface be convex.* By enforcing an integrated form of the convexity inequality throughout the passage of the shock, we show in Section 3 that jumps in specific projections of the stress tensor (such as the deviatoric stress) can be realized only for a *sufficiently large* amount of non-normality. As a specific example, we apply the results to a non-associative constitutive law used recently for an analysis of dynamic, steady-state crack growth in plane strain; we find that a discontinuous stress field (with continuous strain) is impossible *unless* the non-normality is sufficiently strong.

Following this main result, we present in Section 4 a more conventional spectral shock wave analysis, but for a very general class of non-associative rate-independent materials that possess plastic compressibility, pressure dependence of yield, and/or dependence of the elastic moduli on previous plastic strain. We simplify and extend existing work on such materials (e.g. MANDEL, 1963; LORET and HARIRECHE, 1991; OTTOSEN and RUNESSON, 1991), by using the Sherman–Morrison formula to obtain

a *simple* closed form solution for the plastic wave speeds in terms of the elastic eigensystem (which may be assumed to be known *a priori*) and two easily calculated scalar properties of the tangent stiffness tensor. If the elastic part of the response is linear and isotropic, we confirm the known result that one plastic wave speed is always equal to the elastic shear wave speed, while the other two plastic wave speeds depend on the stress state within the shock transition layer. By requiring that each stress state in the transition layer permit the actual wave speed, we derive a requirement for permissible paths of stress within the transition layer. Noting that the governing eigenproblem for moving shock waves is identical in form to the eigen-problem for acceleration waves, we provide very simple necessary and sufficient conditions for one of the wave speeds to be zero (*stationary discontinuity*), or for two of the eigenvalues to be complex conjugate (*flutter instability*); these conditions correspond to a change from hyperbolicity to ellipticity (or vice versa, depending on the problem) of the governing equations, which in turn affects computation procedures. We classify the real eigenvalues according to their numerical sign, and their ordering with respect to the elastic eigenvalues. The solution (which includes eigenvectors that have been missed in previous work) is presented in a form that provides insightful comparisons with the eigenvectors for elastic waves. The comprehensiveness of the analysis and the compact, lucid structure of the solutions distinguish this work from related analyses in the literature.

The paper concludes with a demonstration that dependence of the elastic moduli on plastic strain may be treated as an effective non-normality of the plastic strain increment to the yield surface. Such coupling generally makes the tangent stiffness tensor non-self-adjoint *even when the plastic strain increment is normal to the yield surface*, which is a caveat for HILL's (1968) finding that, in the absence of coupling, normality imples self-adjointness. Referring to MRÓZ and RANIECKI's (1976) thermomechanical flow law, TVERGAARD (1982) noted that coupling in elastic–plastic bodies often leads to non-self-adjointness of the tangent stiffness tensor. This non-self-adjointness may sometimes be regarded as an *effective* non-normality of the plastic strain increment to the yield surface. Effective non-normality caused by elastic–plastic coupling has also been noted by MAIER and HUECKEL (1979). We derive explicit forms for the effective non-normal part of the strain increment using the porous elastic moduli given by ZHAO *et al.* (1989).

2. JUMP CONDITIONS

The hypothesized discontinuity surface is taken to be generally curved and propagating with a normal speed V relative to a fixed observer. The shock propagation speed c relative to an observer moving with a material particle at the jump surface is then

$$c = V - \mathbf{v}\cdot\mathbf{n}, \tag{2.1}$$

where $\mathbf{v}$ is the particle velocity and $\mathbf{n}$ is the unit normal of the shock surface pointing in the propagation direction. Here and throughout this paper, a single dot product

represents the inner product between adjacent vectors (so that, in rectangular Cartesian components, adjacent indices are summed).

The region just ahead of the shock is called the (+) side, and the region just behind is called the (−) side. The jump in any field variable a is denoted by double brackets defined as follows:

$$[\![a]\!] \equiv a^+ - a^-, \tag{2.2}$$

where a^+ is the limiting value of a at the (+) side of the shock, and a^- is the value of a at the (−) side.

2.1. *Jump equations*

A requirement that cracking or interpenetration of material not occur is sufficient to ensure continuity of displacement components *normal*, but not tangent, to a shock. DRUGAN and RICE (1984) noted that the physical requirement of finite plastic work production for finite motion of a shock requires the entire material displacement vector $\mathbf{u}$ to be continuous across the shock:

$$[\![\mathbf{u}]\!] = \mathbf{0}. \tag{2.3}$$

Conservation of mass leads to (e.g. CHADWICK, 1976)

$$[\![\rho c]\!] = 0, \tag{2.4}$$

where ρ is the density. Assuming the gradient, $\mathrm{d}\mathbf{u}/\mathrm{d}\mathbf{X}$, of displacement with respect to reference position, $\mathbf{X}$, exists in a neighborhood of the shock and tends to finite (+) side and (−) side limits as the shock is approached, it can be shown (e.g. HILL, 1961) that the jump in the displacement gradient is of the form

$$[\![\partial\mathbf{u}/\partial\mathbf{X}]\!] = \boldsymbol{\lambda}\mathbf{N}, \tag{2.5}$$

where $\boldsymbol{\lambda}$ is a vector called the "characteristic segment", $\mathbf{N}$ is the unit normal to the image of the discontinuity surface in the reference configuration, and $\boldsymbol{\lambda}\mathbf{N}$ is a dyad. For small displacement gradients, $\mathbf{N} \approx \mathbf{n}$.

By taking the time derivative of displacement following the shock, enforcing (2.3), it can be shown (HADAMARD, 1903) that the jump in velocity is related to the jump in displacement gradient by

$$[\![\mathbf{v}]\!] = -c_0[\![\partial\mathbf{u}/\partial\mathbf{X}]\!]\cdot\mathbf{N} = -c_0\boldsymbol{\lambda} \approx -c\boldsymbol{\lambda}, \tag{2.6}$$

having applied (2.5). Here, c_0 is the speed of the discontinuity surface in the reference configuration, and $c_0 \approx c$ for small displacement gradients.

For dynamic deformations, conservation of linear momentum leads to the well-known (e.g. CHADWICK, 1976) jump equation

$$\mathbf{n}\cdot[\![\boldsymbol{\sigma}]\!] = -\rho c[\![\mathbf{v}]\!] = +\rho c c_0\boldsymbol{\lambda} \approx \rho c^2\boldsymbol{\lambda}, \tag{2.7}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress. Importantly, because ρc^2 is an approximation to $\rho c c_0$, and because $[\![c_0]\!] = 0$, (2.4) permits us to treat ρc^2 as constant across the shock even though c is not constant.

For small displacement gradients, the jump in the total strain $\boldsymbol{\varepsilon}$ is approximately

the symmetric part of the jump in the displacement gradient. Therefore, recalling (2.5) and (2.7), the jump in infinitesimal strain is given by

$$[\![\boldsymbol{\varepsilon}]\!] \approx \operatorname{sym}(\mathbf{n}\boldsymbol{\lambda}) = \frac{1}{\rho c^2} \operatorname{sym}(\mathbf{n}[\![\mathbf{T}]\!]) = \frac{1}{\rho c^2} \mathbb{Q} : [\![\boldsymbol{\sigma}]\!], \tag{2.8}$$

where $\mathbf{T} \equiv \mathbf{n} \cdot \boldsymbol{\sigma}$, and $\mathbb{Q}$ is a non-invertible fourth-order linear operator defined such that for any second-order tensor $\mathbf{A}$,

$$\mathbb{Q} : \mathbf{A} = \operatorname{sym}(\mathbf{n}\mathbf{n} \cdot \operatorname{sym} \mathbf{A}). \tag{2.9}$$

In Cartesian components, $Q_{ijrs} = \frac{1}{4}(n_i\delta_{jr}n_s + n_i\delta_{js}n_r + n_j\delta_{ir}n_s + n_j\delta_{is}n_r)$, where δ_{ij} is the Kronecker delta. Here and throughout this paper, "sym" denotes the symmetric part, and a double dot product (:) represents the tensor inner product between adjacent dyads [so that, in Cartesian components, adjacent indices are summed *pairwise*—e.g. $(\mathbb{Q} : \mathbf{A})_{ij} = Q_{ijrs}A_{rs}$].

We note for future reference that

$$[\![\boldsymbol{\varepsilon}]\!] : [\![\boldsymbol{\sigma}]\!] = \frac{1}{\rho c^2} [\![\boldsymbol{\sigma}]\!] : \mathbb{Q} : [\![\boldsymbol{\sigma}]\!] = \rho c^2 \boldsymbol{\lambda} \cdot \boldsymbol{\lambda}. \tag{2.10}$$

2.2. *Incremental forms of the jump equations*

COURANT and FRIEDRICHS (1948) proved in the gas dynamics context that a shock may be viewed as the limit as thickness vanishes of a narrow transition layer in which field quantities vary continuously as if in a simple wave. As pointed out by LEIGHTON *et al.* (1987), their proof also shows that the *sequence* of states in the transition layer must also be the same as if the transition had occurred in a simple wave. That is, if the jump in some (scalar or tensor) quantity is zero across the transition layer, then that quantity is approximately constant in space and time for all points along a direct path through the transition layer. COURANT and FRIEDRICHS' (1948) conclusions result from their demonstration that entropy changes across shocks can be disregarded for all but the strongest shocks. DRUGAN and SHEN (1990) argued that similar conclusions are sensible for elastic–plastic solids because (WALLACE, 1980) entropy changes across shocks in such materials can be sensibly neglected for a significant range of shock strengths.

Recall that ρc^2 is an approximation to $\rho c c_0$, and so may be treated as constant in the transition layer. Hence, the jump equations (2.7) and (2.8) may be written in the following incremental forms that constrain stress and deformation paths experienced by a particle during shock passage:

$$\mathbf{n} \cdot \mathrm{d}\boldsymbol{\sigma} = -\rho c \, \mathrm{d}\mathbf{v}, \tag{2.11}$$

$$\mathrm{d}\boldsymbol{\varepsilon} = \frac{1}{\rho c^2} \operatorname{sym}(\mathbf{n} \, \mathrm{d}\mathbf{T}) = \frac{1}{\rho c^2} \mathbb{Q} : \mathrm{d}\boldsymbol{\sigma}. \tag{2.12}$$

Importantly, these increments apply not only spatially, but also as increments following a material particle through the shock; hence, the increments $\mathrm{d}\boldsymbol{\varepsilon}$ and $\mathrm{d}\boldsymbol{\sigma}$ are further constrained by the constitutive law.

3. Discontinuity Restrictions for a General Class of Non-classical Materials

3.1. *Non-associative constitutive formulation*

Assuming small displacement gradients, the total strain increment may be additively decomposed into elastic and plastic parts:

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^{e} + d\boldsymbol{\varepsilon}^{p}. \tag{3.1}$$

The yield surface is defined as the boundary in six-dimensional symmetric tensor space of the set of stresses achievable from the current stress $\boldsymbol{\sigma}_A$ via a reversible deformation path. The yield surface depends, in general, on the currently applied stress $\boldsymbol{\sigma}_A$ and on the previous plastic strain history. A yield function Φ is defined according to:

$$\Phi(\boldsymbol{\sigma}; \boldsymbol{\sigma}_A, \alpha_1, \ldots, \alpha_K) \begin{cases} < 0 & \text{for all } \boldsymbol{\sigma} \text{ within the yield surface} \\ = 0 & \text{for all } \boldsymbol{\sigma} \text{ on the yield surface} \\ > 0 & \text{for all other } \boldsymbol{\sigma} \end{cases}, \tag{3.2}$$

where the α_is represent parameters that depend on the plastic strain history. The elastic stress set is assumed convex; i.e.

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{o}) : \mathbf{m} \geqslant 0 \quad \text{for all } \boldsymbol{\sigma} \text{ and } \boldsymbol{\sigma}^{o} \text{ satisfying } \Phi(\boldsymbol{\sigma}) = 0 \quad \text{and} \quad \Phi(\boldsymbol{\sigma}^{o}) \leqslant 0, \tag{3.3}$$

where $\mathbf{m}$ is the outward unit normal to the yield surface at $\boldsymbol{\sigma}$ or, if $\boldsymbol{\sigma}$ is at a vertex, $\mathbf{m}$ is any member of the cone of limiting normals. As Hill (1968) has observed, convexity of the yield surface seems to be implied by many sets of experimental data and, to our knowledge, no data have conclusively suggested otherwise.

Yield surface vertices, pressure sensitivity of yield and non-normality of inelastic strain increments to the yield surface have been observed experimentally (with varying degrees of certainty), and they have been used analytically to predict, for example, bifurcation phenomena such as flow localization. To account approximately for vertex effects in localization, Storen and Rice (1975) regard the yield surface as *smooth*, with the plastic strain increment non-normal to the yield surface.

The plastic strain increment can always be decomposed into parts normal and tangent to the yield surface:

$$d\boldsymbol{\varepsilon}^{p} = d\gamma \mathbf{m} + d\boldsymbol{\varepsilon}^{pt}. \tag{3.4}$$

Here, $d\gamma$ is the component of the plastic strain increment in the direction of the unit normal (or relevant member of the cone of limiting normals) to the yield surface (i.e. $d\gamma = d\boldsymbol{\varepsilon}^{p} : \mathbf{m}$), and $d\boldsymbol{\varepsilon}^{pt}$ is the part of the plastic strain increment tangent to the yield surface. Strictly speaking, $d\gamma$ is not a material increment, but is written so as a notational convenience.

For rate-independent materials, Euler's theorem for homogeneous functions guarantees the existence of a unique fourth-order tensor $\mathbb{M}_1$ such that

$$d\boldsymbol{\varepsilon}^{pt} = \mathbb{M}_1 : d\boldsymbol{\sigma}. \tag{3.5}$$

For non-hardening materials, the existence of $\mathbb{M}_1$ is an assumption (by definition of $d\boldsymbol{\varepsilon}^{pt}$, $\mathbb{M}_1$ will satisfy $\mathbf{m} : \mathbb{M}_1 : d\boldsymbol{\sigma} = 0$). This flow law permits pressure sensitivity of yield

and arbitrary direction of non-normality of the plastic strain increment. Although in general $\mathbb{M}_1$ might be homogeneous of degree zero in stress increment and arbitrarily dependent on stress, plastic strain history, or any other relevant parameters, we will assume $\mathbb{M}_1$ is constant within the shock transition layer; i.e. $[\![\mathbb{M}_1]\!] \approx \mathbb{0}$. This assumption is motivated by a similar assumption in the specific flow law to be analyzed in Section 3.3.

We assume $\mathrm{d}\gamma \geqslant 0$ (i.e. the plastic strain increment is directed to the *outside* of the yield surface); in some circumstances (such as in the example of Section 3.3), an assumption of non-negative plastic work rate can be used to *prove* $\mathrm{d}\gamma \geqslant 0$. Because $\mathrm{d}\boldsymbol{\sigma}$ and $\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}}$ are symmetric, $\mathbb{M}_1$ may be assumed (without loss in generality) to be range- and domain-symmetric; i.e.

$$(\mathbb{M}_1)_{ijrs} = (\mathbb{M}_1)_{jirs} = (\mathbb{M}_1)_{ijsr}. \tag{3.6}$$

We will also assume that $\mathbb{M}_1$ is self-adjoint; i.e.

$$(\mathbb{M}_1)_{ijrs} = (\mathbb{M}_1)_{rsij}. \tag{3.7}$$

Assuming the elastic part of the response is linear, the flow rule for the *total* strain increment is:

$$\mathrm{d}\boldsymbol{\varepsilon} = \mathbb{M}^* : \mathrm{d}\boldsymbol{\sigma} + \mathrm{d}\gamma \mathbf{m}, \tag{3.8}$$

where $\mathbb{M}^* \equiv \mathbb{M}_1 + \mathbb{M}$, and $\mathbb{M}$ is the elastic compliance tensor. Incidentally, this flow law provides a counter-example showing that normality (or the lack of it) cannot be ascertained simply by inspecting the form of the total strain increment flow law because, we note, the flow law (3.8) is *mathematically* identical to conventional (associative) plasticity flow laws—a key difference is that the "pseudo-compliance" $\mathbb{M}^*$ need not be positive definite. We will now exploit this fact to demonstrate that a moving surface of discontinuity in stress is severely restricted whenever $\mathbb{M}^*$ is positive definite, ruling out, in some instances, the very existence of such a surface and, in other instances, bounding the magnitude of the stress jump.

Recalling the assumption that $\mathrm{d}\gamma \geqslant 0$, the statement of convexity (3.3) may be combined with the flow rule (3.8) to give

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\circ}) : (\mathrm{d}\boldsymbol{\varepsilon} - \mathbb{M}^* : \mathrm{d}\boldsymbol{\sigma}) \geqslant 0. \tag{3.9}$$

3.2. *Discontinuity analysis*

Using the jump equation (2.12), the convexity inequality (3.9) may be written within the shock transition layer as

$$-(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\circ}) : \left(\mathbb{M}^* - \frac{1}{\rho c^2} \mathbb{Q} \right) : \mathrm{d}\boldsymbol{\sigma} \geqslant 0. \tag{3.10}$$

Generalizing the key idea of DRUGAN and RICE (1984), we integrate this convexity inequality *at a material particle* as the shock passes

$$-\int_{\sigma^+}^{\sigma} (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\circ}) : \left(\mathbb{M}^* - \frac{1}{\rho c^2} \mathbb{Q} \right) : \mathrm{d}\boldsymbol{\sigma} \geqslant 0. \tag{3.11}$$

By choosing $\boldsymbol{\sigma}^{\circ}$ to be constant in the shock and recalling that $\mathbb{M}^*$ is assumed to be constant throughout the passage of the shock, the integral (3.11) may be evaluated explicitly to yield

$$\left[\!\left[(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\circ}):\left(\mathbb{M}^*-\frac{1}{\rho c^2}\,\mathbb{Q}\right):(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\circ})\right]\!\right] \geqslant 0. \tag{3.12}$$

This result holds for small displacement gradient deformations so long as the chosen $\boldsymbol{\sigma}^{\circ}$ remains on or within the yield surface throughout passage of the shock and the jump of $\mathbb{M}^*$ is negligible. The flow law (3.8) permits any direction of plastic strain increment; however, as will be discussed later, when the plastic strain increment has an isotropic component, the elastic moduli (and, therefore, possibly $\mathbb{M}^*$) cannot in general be reasonably assumed to be constant.

There are flow laws currently used in the literature for which $\mathbb{M}^*$ may be safely assumed to be constant. For example, NEMAT-NASSER and OBATA (1990) use such a flow law in their analysis of steady-state dynamic crack growth (in Section 3.3, their flow law will be used to illustrate the results of this section). As pointed out by NEEDLEMAN and RICE (1978), the difference in yield points in compression and tension observed by SPITZIG *et al.* (1975) may be modeled as a non-normality to a pressure sensitive yield surface, and, according to Spitzig's observations, the plastic strain increment remains nearly deviatoric with negligible dependence of the elastic moduli on plastic straining. Thus, the assumption that $\mathbb{M}^*$ is constant seems reasonable in this case.

Specific choices for $\boldsymbol{\sigma}^{\circ}$. Any choice for $\boldsymbol{\sigma}^{\circ}$ is restricted by the requirement that $\boldsymbol{\sigma}^{\circ}$ be within or on the yield surface for all states throughout the shock transition zone. If $\boldsymbol{\sigma}^{\circ} = \mathbf{0}$ is admissible in this sense, (3.12) gives

$$\left[\!\left[\boldsymbol{\sigma}:\left(\mathbb{M}^*-\frac{1}{\rho c^2}\,\mathbb{Q}\right):\boldsymbol{\sigma}\right]\!\right] \geqslant 0. \tag{3.13}$$

Incidentally, for classical associative plasticity, the choice $\boldsymbol{\sigma}^{\circ} = \mathbf{0}$ would correspond to enforcement of non-negative plastic work rate and so would be an admissible choice even if $\mathbf{0}$ were not in or on the yield surface.

If the choice $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{-}$ is admissible, (3.12) gives, upon rearrangement,

$$[\![\boldsymbol{\sigma}]\!]:\left(\mathbb{M}^*-\frac{1}{\rho c^2}\,\mathbb{Q}\right):[\![\boldsymbol{\sigma}]\!] \geqslant 0, \tag{3.14}$$

or, using the identity (2.10),

$$[\![\boldsymbol{\sigma}]\!]:\mathbb{M}^*:[\![\boldsymbol{\sigma}]\!] \geqslant \rho c^2 \boldsymbol{\lambda}\cdot\boldsymbol{\lambda}. \tag{3.15}$$

If the choice $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{+}$ is admissible, (3.12) gives

$$[\![\boldsymbol{\sigma}]\!]:\mathbb{M}^*:[\![\boldsymbol{\sigma}]\!] \leqslant \rho c^2 \boldsymbol{\lambda}\cdot\boldsymbol{\lambda}. \tag{3.16}$$

As noted by DRUGAN and RICE (1984), the choice $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{+}$ is admissible for any material having the property that its current yield locus at any stage in a deformation

incorporates all prior yield loci. This material class thus includes elastic–plastic behavior characterized by no hardening (i.e. ideal plasticity), isotropic hardening and many types of anisotropic hardening, including many cases of yield surface vertex formation. Similarly, the choice $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{-}$ is admissible for non-hardening materials or for materials that have commenced isotropic softening by the time the shock front arrives.

For some materials (including but not limited to non-hardening materials) the choices $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{+}$ and $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{-}$ are *both* admissible, in which case (3.15) and (3.16) imply

$$[\![\boldsymbol{\sigma}]\!] : \mathbb{M}^* : [\![\boldsymbol{\sigma}]\!] = \rho c^2 \boldsymbol{\lambda} \cdot \boldsymbol{\lambda}. \tag{3.17}$$

The special case of continuous strain or quasi-static deformation. For continuous strain problems, $\boldsymbol{\lambda} \equiv \mathbf{0}$, and for quasi-static problems, $\rho\boldsymbol{\lambda} \approx \mathbf{0}$. In either case, (3.16) becomes

$$[\![\boldsymbol{\sigma}]\!] : \mathbb{M}^* : [\![\boldsymbol{\sigma}]\!] \leqslant 0, \tag{3.18}$$

which (recall) holds provided the choice $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{+}$ is admissible. This inequality is exact for continuous strain deformations and an approximation for quasi-static deformations.

The inequality (3.18) immediately shows that a stress jump is impossible whenever $\mathbb{M}^*$ is positive definite. Thus, recalling (3.5), *a necessary condition for the existence of a stress jump is that the amount of non-normality be sufficiently large* that components of $\mathbb{M}_{\mathrm{I}}$ become sufficiently negative so that (when added to the positive-definite elastic compliance tensor $\mathbb{M}$) the pseudo compliance $\mathbb{M}^*$ is non-positive definite. Consequently, *the previous results of* DRUGAN *and* RICE (1984) *and* DRUGAN *and* SHEN (1987) *are valid for a finite range of general non-normality.*

It may be that $\mathbb{M}^*$ is positive definite only with respect to some *subspace* of the symmetric tensors, in which case the above conclusions may be generalized as follows: suppose that $\mathbb{M}^*$ has the property $\mathbb{M}^* = \mathbb{P}_{\mathscr{A}} : \mathbb{M}^* : \mathbb{P}_{\mathscr{A}}$ for some fourth-order projection operator $\mathbb{P}_{\mathscr{A}}$ onto some specific linear tensor manifold $\mathscr{A}$. Then the inequality (3.18) implies the following stronger existence condition for moving discontinuities in stress with continuous strain and/or quasi-static deformations:

$$[\![\mathbb{P}_{\mathscr{A}} : \boldsymbol{\sigma}]\!] = \mathbf{0} \quad \text{if} \quad \mathbb{M}^* \text{ is positive definite with respect to } \mathscr{A}, \tag{3.19a}$$

which holds provided the choice $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{+}$ is admissible. By "*positive definite with respect to* $\mathscr{A}$," we mean $\mathbf{A} : \mathbb{M}^* : \mathbf{A} > 0$ for all non-zero tensors $\mathbf{A}$ in the manifold $\mathscr{A}$.

Similarly, if the choice $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{-}$ is admissible, then

$$[\![\mathbb{P}_{\mathscr{A}} : \boldsymbol{\sigma}]\!] = \mathbf{0} \quad \text{if} \quad \mathbb{M}^* \text{ is negative definite with respect to } \mathscr{A}. \tag{3.19b}$$

If the choices $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{+}$ and $\boldsymbol{\sigma}^{\circ} = \boldsymbol{\sigma}^{-}$ are *both* admissible, then

$$[\![\mathbb{P}_{\mathscr{A}} : \boldsymbol{\sigma}]\!] = \mathbf{0} \quad \text{if} \quad \mathbb{M}^* \text{ is definite with respect to } \mathscr{A}. \tag{3.19c}$$

Equations (3.19) are the main results of this section. We will now analyze a specific flow law for incompressible plane-strain deformations to illustrate how (3.19) can be used to rule out stress jumps altogether.

3.3. *An illustrative example*

Recently, NEMAT-NASSER and OBATA (1990) proposed a solution for the stress field near the tip of a steadily growing crack in a non-associative, fully incompressible elastic–plastic material for dynamic, plane-strain, small displacement gradient deformations. They claim that the presence of even an infinitesimal amount of non-normality leads to stress—*but not strain*—jump discontinuities, which has been proved impossible by DRUGAN and SHEN (1987) and LEIGHTON *et al.* (1987) whenever normality is assumed from the outset. Nemat-Nasser and Obata suggest that "the near-field solution when the normality rule is imposed at the outset, is an isolated solution which cannot be obtained as a limiting case of the solutions with the yield surface tangential component of the plastic strain rate tending to zero." We now use the results of the previous section to demonstrate that discontinuities of the type described by Nemat-Nasser and Obata are possible only for a *sufficiently large* deviation from normality. An infinitesimal amount of non-normality will not produce the continuous strain with discontinuous stress field that they offer in their solution to the growing crack problem.

In the following analysis, we use precisely the same flow law and assumptions as were used by NEMAT-NASSER and OBATA (1990) [or, for the hardening case, by HORI and NEMAT-NASSER (1989)]. The material is assumed to satisfy the Huber–Mises yield criterion

$$\mathbf{S}:\mathbf{S} = 2k^2, \tag{3.20}$$

where $\mathbf{S}$ is the deviatoric stress tensor, and k is the yield stress in pure shear. This yield surface (being a circular cylinder in six-dimensional symmetric tensor space) is convex. The normal to the yield surface is coaxial with $\mathbf{S}$. For a non-hardening material, k is constant and the yield surface remains fixed in stress space. For an isotropically work hardening material (assuming a non-negative plastic work rate), k increases with plastic deformation. In either case, $\boldsymbol{\sigma}^+$ will—as required for application of (3.19a, c)—remain on or within the yield surface as a particle passes through a shock.

Nemat-Nasser and Obata use *rate* instead of incremental forms of the constitutive laws, so we will follow their conventions in this section. They consider a plastic strain rate of the form

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda\mathbf{S} + \frac{\lambda_1}{k}\dot{\mathbf{S}}, \tag{3.21}$$

where λ_1 and λ are scalars, with λ_1/k assumed constant, and a superimposed dot denotes the material time derivative. For a non-hardening material, continued satisfaction of (3.20) requires that $\mathbf{S}:\dot{\mathbf{S}} = 0$. For an isotropically work hardening material, k increases with plastic deformation, but HORI and NEMAT-NASSER (1989) imply that, to leading order as the crack tip is approached, $\mathbf{S}:\dot{\mathbf{S}} \approx 0$ (see their equation 3.12) and that the plastic strain rate is approximated by (3.21); furthermore, by plotting their results for a given value of their modified Mach number (which depends on λ_1/k) they tacitly assume that λ_1/k is approximately constant. Accepting these claims only for the purpose of contradiction, we conclude that for both non-hardening

and hardening materials, the second term in (3.21) is orthogonal to $\mathbf{S}$ and, therefore, represents the tangential component of the plastic strain rate (at least to leading order for the hardening material). Furthermore, because $\mathbf{S}:\dot{\mathbf{S}} = 0$, an assumption of non-negative plastic work rate *guarantees* that $\lambda \geqslant 0$.

Assuming elastic as well as plastic incompressibility, Nemat-Nasser and Obata find that the flow rule for the *total* strain rate is

$$\dot{\boldsymbol{\varepsilon}} = \frac{3}{2E^*}\dot{\mathbf{S}} + \lambda \mathbf{S}, \tag{3.22}$$

where

$$\frac{1}{E^*} \equiv \frac{1}{E} + \frac{2\lambda_1}{3k}, \tag{3.23}$$

and E is Young's modulus.

This simple material model is a special case of the general model studied in the preceding section. Comparing (3.22) with (3.8), we identify

$$\mathbf{m} = \frac{\mathbf{S}}{\sqrt{\mathbf{S}:\mathbf{S}}} = \frac{\mathbf{S}}{\sqrt{2k}}, \tag{3.24}$$

$$\dot{\gamma} = \lambda\sqrt{\mathbf{S}:\mathbf{S}} = \sqrt{2k}\lambda, \tag{3.25}$$

$$\mathbb{M}^* = \left(\frac{\lambda_1}{k} + \frac{3}{2E}\right)\mathbb{D}_s, \tag{3.26}$$

where $\mathbb{D}_s$ is the fourth-order orthogonal projection operator from nine-dimensional tensor space to the linear manifold, $\mathscr{A}$, of symmetric deviatoric tensors; i.e. for any second-order tensor $\mathbf{A}$,

$$\mathbb{D}_s:\mathbf{A} = \operatorname{sym}\mathbf{A} - \tfrac{1}{3}(\operatorname{tr}\mathbf{A})\mathbf{I}. \tag{3.27}$$

In component form,

$$(\mathbb{D}_s)_{ijrs} = \tfrac{1}{2}(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) - \tfrac{1}{3}\delta_{ij}\delta_{rs}. \tag{3.28}$$

Observe that

$$\mathbb{M}^* = \mathbb{D}_s:\mathbb{M}^*:\mathbb{D}_s, \tag{3.29}$$

and, because $\mathbb{D}_s$ is a projection operator, the premises for application of (3.19) are satisfied.

Using (3.26), we see that for any non-zero, symmetric, deviatoric, second-order tensor $\mathbf{A}$,

$$\mathbf{A}:\mathbb{M}^*:\mathbf{A} = \left(\frac{\lambda_1}{k} + \frac{3}{2E}\right)\mathbf{A}:\mathbf{A} > 0 \quad \text{if} \quad \lambda_1 > -\frac{3k}{2E}. \tag{3.30}$$

Applying (3.19a), we conclude

$$[\![\mathbf{S}]\!] = \mathbf{0} \quad \text{if} \quad \lambda_1 > -\frac{3k}{2E}, \tag{3.31a}$$

and, applying (3.19c) for the non-hardening case,

$$[\![\mathbf{S}]\!] = \mathbf{0} \quad \text{if} \quad \lambda_1 \neq -\frac{3k}{2E}. \tag{3.31b}$$

To this point, we have no information about a jump in pressure. However, (2.8) shows that continuous strain requires continuous traction, which in turn implies that a jump in pressure is necessarily accompanied by a jump in deviatoric stress. Hence,

$$\boxed{[\![\boldsymbol{\sigma}]\!] = \mathbf{0} \quad \text{if} \quad \lambda_1 > -\frac{3k}{2E}}. \tag{3.32}$$

Observe that not only must λ_1 be negative for a stress jump to occur, it must be *sufficiently* negative and, for the non-hardening material, it must be *identically* equal to $-3k/2E$. An *infinitesimal* amount of non-normality will not permit stress jumps with continuous strain in the growing crack stress field.

Small, but not infinitesimal, amounts of non-normality may affect the solution to field equations. Consider, for example, the flow law (3.21) when λ_1 is at its critical negative value, $-3k/2E$:

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \lambda \mathbf{S} - \frac{3}{2}\frac{\dot{\mathbf{S}}}{E}. \tag{3.33}$$

The tangential term is of the order of the stress rate divided by elastic modulus and may, therefore, be neglected except in regions of high stress rates (such as shock transition zones).

4. Improved Spectral Shock Wave Analysis Applied to Non-classical Materials

In this section we perform a more conventional shock wave eigenvalue analysis on an extremely general class of non-conventional rate-independent material models that permit non-normality of the plastic strain increment to the yield surface, plastic compressibility, elastic anisotropy and coupling phenomena such as dependence of the elastic moduli on previous plastic deformation.

The shock wave speeds and direction of the jump in traction are determined by an eigen-problem mathematically identical to the acceleration wave eigen-problem. By employing a very useful tensor identity, we present a complete solution that exhibits an appealing, lucid structure. In the case of elastic isotropy, we derive closed-form solutions for the plastic eigenvalues and eigenvectors in terms of only *two key scalars* which are coefficients in the characteristic polynomial and are easily calculated from the tangent stiffness tensor. It is known that, for elastic isotropy, at least one of the

three plastic eigenvalues always equals the *elastic* shear eigenvalue, while the other two depend on the state in the transition layer. We show that at least one of these "state-dependent" plastic eigenvalues can equal an elastic eigenvalue if and only if one of the two key scalars vanishes. *Simple* necessary and sufficient conditions for the existence of zero or complex-conjugate eigenvalues are derived, and these conditions depend only on the two key scalars. By plotting the two key scalars against each other, we illustrate the locus of points that correspond to double roots, negative roots and lines of constant eigenvalue. This figure also illustrates specific ranges of the two key scalars that will result in any given ordering of the state-dependent plastic eigenvalues with respect to the elastic eigenvalues. Finally, we derive the *complete* set of eigenvectors associated with any given eigenvalue, showing that the eigenvector associated with a *non*-elastic eigenvalue has a very simple structure in terms of the elastic eigen-system.

Novel features (to our knowledge) of the analysis include the generality of the constitutive class analyzed, the lucidity and completeness of the solution, and the expression of the isotropic material results in terms of only two scalars with an associated graphical classification of the eigenvalues.

4.1. *Wave propagation eigen-problem for a general class of arbitrarily anisotropic elastic–plastic flow laws*

The constitutive law considered in this section is more general than the one employed in Section 3. The stress increment $\mathrm{d}\boldsymbol{\sigma}$ is regarded as a function of the strain increment $\mathrm{d}\boldsymbol{\varepsilon}$, stress $\boldsymbol{\sigma}$ and various other parameters $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such as temperature and hardening moduli. The material is assumed to be rate independent; that is, for any scalar s,

$$\mathrm{d}\boldsymbol{\sigma}(\boldsymbol{\sigma}, s\,\mathrm{d}\boldsymbol{\varepsilon}, \alpha_1, \ldots, \alpha_n) = s\,\mathrm{d}\boldsymbol{\sigma}(\boldsymbol{\sigma}, \mathrm{d}\boldsymbol{\varepsilon}, \alpha_1, \ldots, \alpha_n). \tag{4.1}$$

By Euler's theorem for homogeneous functions, there exists a fourth-order tensor $\boldsymbol{\xi}$ such that

$$\mathrm{d}\boldsymbol{\sigma} = \boldsymbol{\xi} : \mathrm{d}\boldsymbol{\varepsilon}, \tag{4.2}$$

where

$$\xi_{ijkl} = \frac{\partial(\mathrm{d}\sigma)_{ij}}{\partial(\mathrm{d}\varepsilon)_{kl}}. \tag{4.3}$$

According to Euler's theorem, the fourth-order tensor $\boldsymbol{\xi}$ could in general be homogeneous of degree zero in $\mathrm{d}\boldsymbol{\varepsilon}$. We will assume that $\boldsymbol{\xi}$ is *independent* of $\mathrm{d}\boldsymbol{\varepsilon}$, but otherwise arbitrarily dependent on strain, plastic strain history, or any other relevant parameters. To our knowledge, most rate-independent flow laws currently in use satisfy this assumption. Because $\mathrm{d}\boldsymbol{\sigma}$ and $\mathrm{d}\boldsymbol{\varepsilon}$ are symmetric, $\boldsymbol{\xi}$ is necessarily range-symmetric and, without loss in generality, domain-symmetric; i.e.

$$\xi_{ijmn} = \xi_{jimn} = \xi_{ijnm}. \tag{4.4}$$

However, ξ is not necessarily self-adjoint (that is, in general $\xi_{ijmn} \neq \xi_{mnij}$). Physical mechanisms such as internal friction or the nucleation of microvoids can result in a non-associative (non-normal) plastic flow law. *Implicitly neglecting coupling effects*, HILL (1968) proved that ξ will be self-adjoint if the plastic strain rate is normal to the yield surface in tensor space, where Hill defines the yield surface as the boundary of the set of stresses achievable from the current stress state via an elastic strain. However, as demonstrated in Section 5, such normality does *not* imply self-adjointness of ξ if one allows coupling effects such as dependence of the elastic moduli on plastic straining. Hence, *an associative plastic flow law is not necessarily self-adjoint, and vice versa.*

Using the jump equation (2.12), the flow law (4.2) becomes

$$\mathrm{d}\boldsymbol{\sigma} = \frac{1}{\rho c^2}(\xi \cdot \mathbf{n}) \cdot \mathrm{d}\mathbf{T}. \tag{4.5}$$

Dotting both sides of this equation by $\mathbf{n}$ leads to the well-known eigen-problem

$$(\mathbf{A} - x\mathbf{I}) \cdot \mathbf{w} = \mathbf{0}, \tag{4.6}$$

where

$$x \equiv \rho c^2, \tag{4.7}$$

$$\mathbf{A} \equiv \mathbf{n} \cdot \xi \cdot \mathbf{n} \tag{4.8}$$

and the eigenvector $\mathbf{w}$ is parallel to the traction increment $\mathrm{d}\mathbf{T}$. Once the eigenvector for the traction increment is found, the associated directions for the stress and strain increments are obtained from (4.5) and (2.12), respectively.

The second-order tensor $\mathbf{A}$ in (4.8) is the same as the so-called plastic acoustic tensor from plane and acceleration wave analysis except that $\mathbf{A}$ depends on the stress and plastic strain states within the shock transition layer. Similar eigen-problems for the wave speeds are reviewed for more specialized constitutive laws by TING (1976).

The wave speeds are guaranteed to be real if ξ is self-adjoint (i.e. if $\xi_{ijmn} = \xi_{mnij}$). However, there are important flow laws in the literature for which ξ is *not* self-adjoint. Specifically, we are interested in flow laws for which the tangent *compliance tensor* ξ^{-1} is of the form

$$\xi^{-1} = \mathbb{M} + \frac{1}{h}\mathbf{PQ}, \tag{4.9}$$

where $\mathbb{M}$ is the fourth-order, self-adjoint, positive-definite, instantaneous elastic compliance tensor, h is a scalar, $\mathbf{P}$ and $\mathbf{Q}$ are symmetric second-order tensors, and $\mathbf{PQ}$ is a tensor–tensor dyad [i.e. in Cartesian components, $(\mathbf{PQ})_{ijmn} = P_{ij}Q_{mn}$]. The elastic compliance tensor $\mathbb{M}$ is arbitrarily anisotropic, and the elastic moduli may permissibly change with plastic strain. The second term in (4.9) characterizes the non-recoverable part of the material response, but no precise physical meaning of h, $\mathbf{P}$ or $\mathbf{Q}$ is invoked in the upcoming analysis (except in examples).

NEEDLEMAN and RICE (1978) point out that the plastic compliance tensor (4.9) has applications to metal plasticity, modeling, for example, the difference in compressive and tensile yield strengths observed by SPITZIG and RICHMOND (1984) for high strength steels, as well as void nucleation in metals. These models are usually used in con-

junction with pressure sensitive yield criteria such as that of GURSON (1977). The flow law (4.9) even includes some modern theories (e.g. PASTOR *et al.*, 1990) that do not employ the concept of a yield surface.

Because $\mathbb{M}$ is self-adjoint, ξ is self-adjoint if and only if $\mathbf{P}$ is coaxial with $\mathbf{Q}$ (i.e. if $\mathbf{P} = \alpha\mathbf{Q}$, where α is a scalar). For *some* constitutive models, $\mathbf{P}$ and $\mathbf{Q}$ are coaxial with the normals to the plastic potential and yield surface respectively; for these models, associativity of the plastic flow law is equivalent to self-adjointness of ξ. However, for many other important materials (such as the coupled material discussed in Section 5), $\mathbf{P}$ and $\mathbf{Q}$ have different physical interpretations; hence, for these models, associativity of the plastic flow law is *not* necessarily equivalent to self-adjointness of ξ.

Inverting (4.9), the tangent stiffness tensor ξ is

$$\xi = \mathbb{E} - \frac{(\mathbb{E}:\mathbf{P})(\mathbf{Q}:\mathbb{E})}{h+\mathbf{Q}:\mathbb{E}:\mathbf{P}}, \tag{4.10}$$

where $\mathbb{E}$ is the fourth-order elastic stiffness tensor (i.e. $\mathbb{E} \equiv \mathbb{M}^{-1}$). The general form (4.10) is well defined even if the inverse ξ^{-1} does not exist (i.e. if $h = 0$).

Applying the definition (4.8), the plastic acoustic tensor $\mathbf{A}$ may be written

$$\mathbf{A} = \mathbf{A}^{e} - \frac{1}{\eta}\mathbf{pq}, \tag{4.11}$$

where $\mathbf{pq}$ is a vector–vector dyad, and

$$\mathbf{A}^{e} \equiv \mathbf{n}\cdot\mathbb{E}\cdot\mathbf{n} \tag{4.12}$$

$$\eta \equiv h+\mathbf{Q}:\mathbb{E}:\mathbf{P} \tag{4.13}$$

$$\mathbf{p} \equiv \mathbf{n}\cdot\mathbb{E}:\mathbf{P} \tag{4.14}$$

$$\mathbf{q} \equiv \mathbf{Q}:\mathbb{E}\cdot\mathbf{n}. \tag{4.15}$$

The elastic acoustic tensor $\mathbf{A}^{e}$ is independent of the stress state within the shock transition layer, but will depend on the plastic strain whenever the elastic moduli change with plastic strain. In general η, $\mathbf{p}$ and $\mathbf{q}$ depend on both the plastic strain history and the stress state within the transition layer. The scalar denominator η is usually positive; for the specific flow law discussed in Section 4.6, h would have to be negative and of the order of elastic moduli in order to make η vanish.

In the case of elastic isotropy, a non-symmetric acoustic tensor like (4.11) has been studied in the pioneering work of MANDEL (1963) and, more recently, by LORET *et al.* (1991, 1990) and OTTOSEN and RUNESSON (1991). The form (4.11) for the plastic acoustic tensor $\mathbf{A}$ is mathematically identical to a tensor studied by TING (1976). However, Ting's tensor corresponds to *classical plasticity* ($\mathbf{P} = \mathbf{Q}$) with *non-classical elasticity* ($M_{ijrs} \neq M_{rsij}$), the latter being at variance with classical thermodynamics. We assume self-adjoint elasticity.

4.2. *Compact closed form solution for the wave speeds*

A closed form solution for the eigenvalues may be readily obtained by noting that for any second-order tensor $\mathbf{B}$ and vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\det(\mathbf{B}+\mathbf{uv}) = \det\mathbf{B}+\mathbf{u}\cdot\mathbf{B}^{c}\cdot\mathbf{v}, \tag{4.16}$$

where $\mathbf{B}^{c}$ is the cofactor, or "adjugate", tensor of $\mathbf{B}$ (i.e. the signed subminors tensor). This identity, which is also known as the Sherman–Morrison formula, follows immediately from the invariant definitions of determinant, trace, and cofactor (e.g. CHADWICK, 1976). Applying (4.16) to (4.6) using (4.11) shows that the eigenvalues are the solutions to

$$\det(\mathbf{A}^{e}-x\mathbf{I}) - \frac{1}{\eta}\,\mathbf{p}\cdot(\mathbf{A}^{\acute{e}}-x\mathbf{I})^{c}\cdot\mathbf{q} = 0. \tag{4.17}$$

Given the self-adjointness and positive definiteness of the elastic stiffness tensor $\mathbb{E}$, the *elastic* acoustic tensor $\mathbf{A}^{e}$ will be symmetric and positive definite and will, therefore, have positive eigenvalues and be diagonal in its principal basis, which may be determined *a priori*. By writing (4.17) in terms of the principal basis of $\mathbf{A}^{e}$, one immediately obtains the characteristic polynomial for the eigenvalues:

$$(x-x_1^{e})(x-x_2^{e})(x-x_3^{e})+\theta_1(x-x_2^{e})(x-x_3^{e})+\theta_2(x-x_1^{e})(x-x_3^{e}) + \theta_3(x-x_1^{e})(x-x_2^{e}) = 0, \tag{4.18}$$

where

$$\theta_1 = \frac{1}{\eta}(\boldsymbol{\delta}_1^{e}\cdot\mathbf{p})(\mathbf{q}\cdot\boldsymbol{\delta}_1^{e}), \quad \text{etc.}, \tag{4.19}$$

and x_i^{e} and $\boldsymbol{\delta}_i^{e}$ are the *elastic* eigenvalues and normalized eigenvectors, respectively. Note that in the limit as $\eta \to \infty$, we recover the three elastic eigenvalues. The characteristic polynomial (4.18) holds for arbitrary plastic and elastic anisotropy and seems to have greater generality and a more compact and lucid form than previous results (cf. LORET *et al.*, 1990; OTTOSEN and RUNESSON, 1991). Equation (4.18) is similar to expressions by MANDEL (1962) and TING (1976) except that these authors consider classical associative flow rules, and Mandel assumed proportional stressing across the shock.

Specialization when the elastic part of the response is isotropic. We now consider the case that the elastic part of the response is both linear and *isotropic* (and remains so even after plastic deformation, though plastic strain is permitted to affect the moduli); then

$$\mathbf{A}^{e} = G\mathbf{I}+(G+\lambda)\mathbf{nn}, \tag{4.20}$$

where G and λ are the Lamé moduli. The corresponding *elastic* eigen-system is

$$x_1^{e} = 2G+\lambda, \quad x_2^{e} = x_3^{e} = G, \quad \boldsymbol{\delta}_1^{e} = \mathbf{n}, \tag{4.21}$$

with the remaining eigenvectors, $\boldsymbol{\delta}_2^{e}$ and $\boldsymbol{\delta}_3^{e}$, being any two perpendicular vectors in the discontinuity surface. To emphasize the structure of the solution to the eigenproblem, we define

$$x_n^{e} \equiv 2G+\lambda, \tag{4.22a}$$

$$x_t^e \equiv G, \tag{4.22b}$$

$$\theta_n \equiv \frac{1}{\eta} p_1 q_1 = \frac{1}{\eta} p_n q_n, \tag{4.23a}$$

$$\theta_t \equiv \frac{1}{\eta}(p_2 q_2 + p_3 q_3) = \frac{1}{\eta}\mathbf{p}_t \cdot \mathbf{q}_t. \tag{4.23b}$$

The subscript "n" stands for the normal projection component (e.g. $p_n \equiv \mathbf{p}\cdot\mathbf{n}$) and the subscript "t" stands for the tangential projection vector (e.g. $\mathbf{p}_t \equiv \mathbf{p} - p_n\mathbf{n}$). We will now show that the complete spectral solution depends on the values of θ_n and θ_t as summarized in Table 1.

Using the above definitions in (4.20) and (4.11), the original eigen-problem (4.6) becomes

$$\left[(x - x_t^e)\mathbf{I} - (x_n^e - x_t^e)\mathbf{nn} + \frac{1}{\eta}\mathbf{pq}\right]\cdot\mathbf{w} = \mathbf{0}, \tag{4.24}$$

and the characteristic polynomial becomes

$$(x - x_t^e)[(x - x_n^e)(x - x_t^e) + \theta_n(x - x_t^e) + \theta_t(x - x_n^e)] = 0, \tag{4.25}$$

which has a structure similar to the general polynomial (4.18).

Note that whenever there is elastic isotropy, one plastic wave speed is *always* the elastic shear wave speed. The quadratic formula gives the remaining two "*state-dependent*" eigenvalues:

$$x = \tfrac{1}{2}[(x_n^e + x_t^e) - (\theta_n + \theta_t)] \pm \tfrac{1}{2}\sqrt{(\theta_n + \theta_t)^2 - 2(\theta_n - \theta_t)(x_n^e - x_t^e) + (x_n^e - x_t^e)^2}. \tag{4.26}$$

OTTOSEN and RUNESSON (1991) recently presented an exact solution for *acceleration* wave speeds in elastic–plastic materials with isotropic elastic response. Their solution

TABLE 1. *Complete spectral solution*

$\bar{\theta}_t$	$\bar{\theta}_n$	Eigen-pairs		
0	0	x_t^e (4.43)	x_t^e (4.43)	x_n^e (4.47)
0	1	x_t^e (4.42)	x_t^e (4.42)	x_t^e (4.42)
0	$\neq 1, 0$	x_t^e (4.43)	x_t^e (4.43)	$x_n - \theta_n$ (4.41)
-1	0	x_t^e (4.45)	x_n^e (4.46)	x_n^e (4.46)
$\neq -1, 0$	0	x_t^e (4.45)	x_n^e (4.47)	$x_t^e - \theta_t$ (4.41)
$\neq 0$	$\neq 0$	x_t^e (4.45)	NE (4.41)	NE (4.41)

The over-bar denotes division by $G + \lambda$. In parentheses is the equation number for the eigenvector, and "NE" means the eigenvalue [given by (4.26)] is not equal to an elastic eigenvalue. The last two columns correspond to the "state-dependent" eigenvalues.

(derived using a less direct method) is consistent with ours, but has a less compact and transparent structure and might, therefore, be less convenient for general applications. The solution (4.26) may also be applied to the multi-phase material with incompressible constituents of LORET and HARIRECHE (1991) if x_n^e and θ_n are replaced by x_n^e/r and θ_n/r, respectively, where r is defined in their equation (6.8).

Non-dimensionalization of the eigen-problem with elastic isotropy. Specific properties of the eigenvalues (such as their numerical sign and ordering relative to the elastic eigenvalues) are most easily derived by using a non-dimensionalized version of the eigen-problem. For any scalar or arbitrary order tensor Z, define an over-bar by

$$\bar{Z} \equiv \frac{Z}{x_n^e - x_t^e} = \frac{Z}{G+\lambda}. \tag{4.27}$$

From (4.22),

$$\bar{x}_t^e = \frac{G}{G+\lambda} = 1-2\nu \quad \text{and} \quad \bar{x}_n^e = \frac{2G+\lambda}{G+\lambda} = 2(1-\nu), \tag{4.28}$$

where ν is Poisson's ratio. The eigen-problem (4.24) takes the following non-dimensional form:

$$\left[(\bar{x}-\bar{x}_t^e)\mathbf{I} - \mathbf{nn} + \frac{1}{\bar{\eta}}\bar{\mathbf{p}}\bar{\mathbf{q}}\right]\cdot\mathbf{w} = \mathbf{0}, \tag{4.29}$$

and, from (4.26), the solution for the non-dimensionalized eigenvalues is

$$\bar{x} = \tfrac{1}{2}[(\bar{x}_n^e + \bar{x}_t^e) - (\bar{\theta}_n + \bar{\theta}_t)] \pm \tfrac{1}{2}\sqrt{(\bar{\theta}_n + \bar{\theta}_t)^2 - 2(\bar{\theta}_n - \bar{\theta}_t) + 1}. \tag{4.30}$$

Noting that $\bar{x}_n^e - \bar{x}_t^e = 1$, we see that $\bar{x} - \bar{x}_t^e$ and $\bar{x} - \bar{x}_n^e$ are each pure functions of $\bar{\theta}_t$ and $\bar{\theta}_n$, which will be later exploited to deduce the ordering of the plastic eigenvalues with respect to the elastic eigenvalues.

Properties of the eigenvalues. The eigenvalues are real only if the discriminant in (4.30) is non-negative. Figure 1 illustrates the (tilted) parabola for which the discriminant vanishes. A sufficient condition for real eigenvalues is $\theta_n \leqslant 0$ and/or $\theta_t \geqslant 0$, or, equivalently,

$$\theta_n - \theta_t \leqslant |\theta_n + \theta_t|. \tag{4.31}$$

Recall that if $\mathbf{P}$ and $\mathbf{Q}$ are known, then θ_n and θ_t are known. When the elastic part of the response is isotropic, (4.14) and (4.15) give

$$\mathbf{p} = 2G\mathbf{n}\cdot\mathbf{P} + \lambda\mathbf{n}\,\mathrm{tr}\,\mathbf{P} = 2G\mathbf{n}\cdot\mathbf{P}^d + K\mathbf{n}\,\mathrm{tr}\,\mathbf{P}, \tag{4.32a}$$

$$\mathbf{q} = 2G\mathbf{n}\cdot\mathbf{Q} + \lambda\mathbf{n}\,\mathrm{tr}\,\mathbf{Q} = 2G\mathbf{n}\cdot\mathbf{Q}^d + K\mathbf{n}\,\mathrm{tr}\,\mathbf{Q}, \tag{4.32b}$$

and (4.13) gives

$$\eta = h + 2G\mathbf{P}^d : \mathbf{Q}^d + K(\mathrm{tr}\,\mathbf{P})(\mathrm{tr}\,\mathbf{Q}), \tag{4.32c}$$

where K is the bulk modulus, "tr" denotes the trace, and a superscript "d" denotes

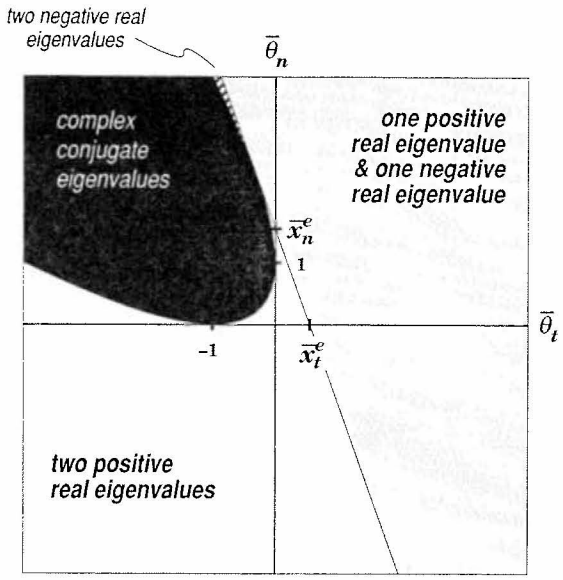

FIG. 1. State-dependent eigenvalues for elastic isotropy. Points on the parabola correspond to a vanishing discriminant in (4.30) and, therefore, to double roots. The slanted line tangent to the parabola corresponds to the situations in which at least one eigenvalue is zero.

the deviatoric part. Thus, using an orthonormal basis having the 1-direction aligned with **n**,

$$\theta_n = \frac{1}{\eta}(2GP^d_{11} + K \text{ tr } \mathbf{P})(2GQ^d_{11} + K \text{ tr } \mathbf{Q}), \tag{4.33a}$$

$$\theta_t = \frac{4G^2}{\eta}(\mathbf{P}^d_{12}\mathbf{Q}^d_{12} + \mathbf{P}^d_{13}\mathbf{Q}^d_{13}). \tag{4.33b}$$

When the constitutive law is self-adjoint (i.e. when **P** is coaxial with **Q**), both θ_t and θ_n will have the same sign and, referring to Fig. 1, the eigenvalues will be real (as expected). Observe that $\theta_t = 0$ whenever **n** is a principal direction of either **P** or **Q**, which confirms a weaker sufficient condition by OTTOSEN and RUNESSON (1991) given in their Table 1. Whenever $\eta > 0$ and the second-order tensors **P** and **Q** share the same deviatoric part, (4.33b) shows that $\theta_t \geqslant 0$, thereby guaranteeing real eigenvalues (see Fig. 1), which was also noted for single phase materials by LORET *et al.* (1990). This result is in agreement with the stronger result by OTTOSEN and RUNESSON (1991) that (in the present notation) $\theta_t \geqslant 0$ whenever $\eta > 0$ and **P** and **Q** share both the same principal directions *and* the same ordering of the eigenvalues.

For the multi-phase material with incompressible constituents discussed by LORET and HARIRECHE (1991), recall that the eigenvalue solution may be obtained by replacing x^e_n and θ_n by x^e_n/r and θ_n/r, respectively. If r is large enough, the denominator in the normalization equation (4.27) could be negative, thereby placing the complex parabola in the *fourth* instead of the second quadrant in Fig. 1, and $\theta_t \geqslant 0$ would be

insufficient to rule out complex eigenvalues. The inequality guaranteeing a non-negative normalization factor can readily be shown to be equivalent to the inequality $\tau \geqslant 0$ derived by LORET and HARIRECHE (1991), where τ is defined in their equation (6.17). In the remainder of the present analysis, we will be concerned with a single constituent so that the complex parabola will always be in the second quadrant as illustrated in Fig. 1.

The critical condition for localization to occur in a homogeneous, uniformly deformed body is known to correspond to the vanishing of a wave speed [e.g. MANDEL (1964) and RICE (1976)]. From (4.25), at least one eigenvalue will be zero if and only if

$$\frac{\theta_n}{x_n^e} + \frac{\theta_t}{x_t^e} = 1, \tag{4.34}$$

which is a straight line in the $\bar{\theta}_n$ vs $\bar{\theta}_t$ plane. Figure 1 divides the real eigenvalue region according to the sign of the state-dependent eigenvalues. The line tangent to the parabola describes the set of $(\bar{\theta}_t, \bar{\theta}_n)$ values for which at least one eigenvalue is zero, with two zero eigenvalues being possible only at the point of tangency.

The ordering of the eigenvalues with respect to the elastic eigenvalues may be determined by regarding the characteristic polynomial (4.25) as a function of θ_n and θ_t with x fixed. It is straightforward to show that lines of constant eigenvalue are straight lines tangent to the complex parabola. As illustrated in Fig. 2, each point in the interior of the real eigenvalue region is intersected by two such lines, one for each state-dependent eigenvalue. Any line tangent to the complex parabola at the part marked "$x < x_t^e$" is a line of constant eigenvalue of magnitude less than the elastic shear eigenvalue and, similarly, any line tangent to the part marked "$x > x_n^e$" corresponds to an eigenvalue greater than the elastic longitudinal eigenvalue. Any line tangent to the part of the parabola between the points $(0, 1)$ and $(-1, 0)$ has an

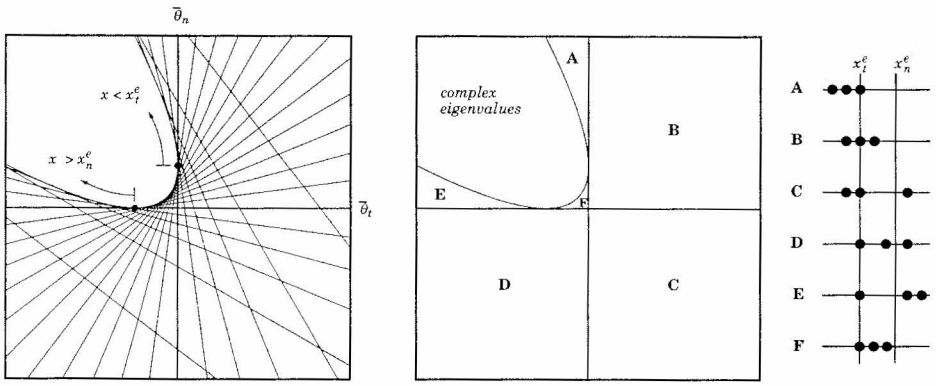

FIG. 2. Eigenvalue ordering for elastic isotropy. Lines of constant eigenvalue are tangent to the parabola. The number lines at the right show the ordering of the plastic eigenvalues relative to the elastic eigenvalues (at least one eigenvalue always equals the elastic shear eigenvalue). Assuming $\eta > 0$, all self-adjoint constitutive laws will lie in region **B**, and constitutive laws for which the deviators of **P** and **Q** are equal will lie in regions **B** and **C**.

eigenvalue x satisfying $x_t^e < x < x_n^e$. The ordering of the eigenvalues relative to the elastic eigenvalues is illustrated by the number lines in Fig. 2. Our ordering for classical self-adjoint constitutive laws (region **B** in Fig. 2) agrees with that derived by MANDEL (1962). Non-self-adjoint constitutive laws will in general have different ordering. Our ordering for deviatoric associativity (regions **B** and **C** in Fig. 2) agrees with that reported by LORET *et al.* (1990).

Conditions for a plastic eigenvalue to equal an elastic eigenvalue. Determination of eigenvectors depends on whether the plastic eigenvalue is equal to an elastic eigenvalue, and, if it is, on its algebraic multiplicity. Whenever $\bar{\theta}_n$ and $\bar{\theta}_t$ are *both* non-zero, neither of the state-dependent eigenvalues given by (4.26) will equal an elastic eigenvalue. Otherwise, at least one of the state-dependent eigenvalues will equal an elastic eigenvalue as outlined below.

(1) At least one state-dependent plastic eigenvalue will equal the elastic longitudinal eigenvalue x_n^e if and only if $\theta_n = 0$, in which case the state-dependent eigenvalues are given by

$$\bar{x} = \begin{cases} \bar{x}_n^e \\ \bar{x}_t^e - \bar{\theta}_t \end{cases} = \begin{cases} 2(1-\nu) \\ (1-2\nu) - \bar{\theta}_t \end{cases}. \tag{4.35}$$

The algebraic multiplicity of x_n^e is *two* at the point $(-1, 0)$ in Fig. 1, and *unity* elsewhere on the $\theta_n = 0$ axis (see the first, fourth and fifth rows in Table 1).

(2) At least one state-dependent plastic eigenvalue will equal the elastic tangential eigenvalue x_t^e if and only if $\theta_t = 0$, in which case the state-dependent eigenvalues are given by

$$\bar{x} = \begin{cases} \bar{x}_n^e - \bar{\theta}_n \\ \bar{x}_t^e \end{cases} = \begin{cases} 2(1-\nu) - \bar{\theta}_n \\ (1-2\nu) \end{cases}. \tag{4.36}$$

The algebraic multiplicity of x_t^e is *three* at the point $(0, 1)$ in Fig. 1, and *two* elsewhere on the $\theta_t = 0$ axis (see the first, second and third rows in Table 1). As mentioned earlier, a sufficient (but not necessary) condition for $\theta_t = 0$ is that $\mathbf{n}$ be a principal direction of either $\mathbf{P}$ or $\mathbf{Q}$.

4.3. *Eigenvectors—arbitrary elastic anisotropy*

To find the eigenvectors, we return to the original eigen-problem (4.6), which, using (4.11), we rewrite as

$$(x\mathbf{I} - \mathbf{A}^e) \cdot \mathbf{w} + \frac{1}{\eta} \mathbf{p}(\mathbf{q} \cdot \mathbf{w}) = \mathbf{0}. \tag{4.37}$$

In terms of the elastic principal basis, this equation may be decomposed into the following system:

$$(x - x_1^e) w_1 + \frac{1}{\eta} p_1 (q_1 w_1 + q_2 w_2 + q_3 w_3) = 0, \tag{4.38a}$$

$$(x-x_2^e)w_2+\frac{1}{\eta}p_2(q_1w_1+q_2w_2+q_3w_3)=0, \tag{4.38b}$$

$$(x-x_3^e)w_3+\frac{1}{\eta}p_3(q_1w_1+q_2w_2+q_3w_3)=0. \tag{4.38c}$$

Thus, the eigenvector associated with a *non*-elastic eigenvalue is

$$\boxed{\mathbf{w}=\alpha\left(\frac{\mathbf{p}_1}{x-x_1^e}+\frac{\mathbf{p}_2}{x-x_2^e}+\frac{\mathbf{p}_3}{x-x_3^e}\right)}, \tag{4.39}$$

where $\mathbf{p}_i$ is the projection of $\mathbf{p}$ onto the ith elastic eigenvector, and α is an arbitrary scalar (chosen to normalize $\mathbf{w}$ if so desired). Although the constitutive tensor $\mathbf{Q}$ does not appear explicitly, it does affect the plastic eigenvalue x. The geometric multiplicity of a non-elastic eigenvalue (i.e. the number of associated eigenvectors) is always unity, even if the algebraic multiplicity is greater than unity.

The solution to (4.38) complicates considerably whenever the eigenvalue is equal to an elastic eigenvalue. For example, if $x=x_1^e$, satisfaction of (4.38a) depends on whether or not p_1 is zero, and for each of these cases, the solution further depends on whether x_2^e or x_3^e equals x_1^e and on the algebraic multiplicity of the eigenvalue x. Each of these possible sub-cases will now be analyzed for the simpler case that the elastic part of the response is isotropic.

4.4. *Eigenvectors—elastic isotropy*

When the elastic part of the elastic–plastic response is isotropic and linear, the eigen-problem (4.29) may be written as the following system:

$$(\bar{x}-\bar{x}_t^e)\mathbf{w}_t+\frac{1}{\bar{\eta}}\bar{\mathbf{p}}_t(\bar{q}_n w_n+\bar{\mathbf{q}}_t\cdot\mathbf{w}_t)=\mathbf{0}, \tag{4.40a}$$

$$(\bar{x}-\bar{x}_n^e)w_n+\frac{1}{\bar{\eta}}\bar{p}_n(\bar{q}_n w_n+\bar{\mathbf{q}}_t\cdot\mathbf{w}_t)=0, \tag{4.40b}$$

where the subscript "n" stands for the normal projection component (e.g. $p_n\equiv\mathbf{n}\cdot\mathbf{p}$) and "t" stands for the tangential projection vector (e.g. $\mathbf{p}_t\equiv\mathbf{p}-p_n\mathbf{n}$).

The eigenvector (4.39) associated with a *non*-elastic eigenvalue can be written in the case of elastic isotropy as

$$\boxed{\mathbf{w}=\alpha\left(\frac{\mathbf{p}_n}{x-x_n^e}+\frac{\mathbf{p}_t}{x-x_t^e}\right)}, \tag{4.41}$$

where $\mathbf{p}_n$ and $\mathbf{p}_t$ are the projections of $\mathbf{p}$ normal to and tangent to the jump surface, respectively, and α is an arbitrary scalar.

Whenever *both* state-dependent eigenvalues are non-elastic (last row in Table 1), it is straightforward to show that the two associated eigenvectors will be perpendicular to each other if and only if

$$\frac{p_{\mathrm{n}}^2}{\theta_{\mathrm{n}}} = \frac{p_{\mathrm{t}}^2}{\theta_{\mathrm{t}}},$$

where p_{n} and p_{t} are the magnitudes of $\mathbf{p}_{\mathrm{n}}$ and $\mathbf{p}_{\mathrm{t}}$, respectively, neither of which [recalling (4.35) and (4.36)] is zero when *both* state-dependent eigenvalues are non-elastic.

Eigenvectors associated with *elastic* eigenvalues depend not only on the nature of $\mathbf{p}$ and $\mathbf{q}$, but also on the algebraic multiplicity of the eigenvalue. That is, the points $(0,1)$ and $(-1,0)$ in Fig. 1 must be treated as special cases.

Consider first the case that $x = x_{\mathrm{t}}^{\mathrm{e}}$. It is shown in the Appendix that if $x = x_{\mathrm{t}}^{\mathrm{e}}$ with an algebraic multiplicity of *three* (second row in Table 1), the solution for the eigenvector $\mathbf{w}$ depends on the tangential projections of $\mathbf{p}$ and $\mathbf{q}$ as follows:

	$\mathbf{p}_{\mathrm{t}} = \mathbf{0}$	$\mathbf{p}_{\mathrm{t}} \neq \mathbf{0}$
$\mathbf{q}_{\mathrm{t}} = \mathbf{0}$	Three eigenvectors, arbitrary direction	Two eigenvectors $\mathbf{n}\cdot\mathbf{w} = 0$
$\mathbf{q}_{\mathrm{t}} \neq \mathbf{0}$	Two eigenvectors $\bar{\mathbf{q}}_{\mathrm{t}}\cdot\mathbf{w} = 0$	One eigenvector $\mathbf{w} = \alpha\bar{\mathbf{p}}_{\mathrm{t}}$

(4.42)

where α is an arbitrary scalar. OTTOSEN and RUNESSON (1991) showed that if the tensors $\mathbf{P}$ and $\mathbf{Q}$ share both the same principal directions and *ordering* of the eigenvalues, then only the upper-left solution is possible. This conclusion can be readily verified for the special case that $\mathbf{P}$ and $\mathbf{Q}$ share the same (or coaxial) deviatoric parts because, in this special case, (4.33b) shows that θ_{t} can be zero [as required by (4.36)] only if the tangential projections of $\mathbf{p}$ and $\mathbf{q}$ *both* vanish. In general, however, if the Ottosen–Runesson conditions do not hold, the geometric multiplicity of the eigenvalue can be less than the algebraic multiplicity.

If $x = x_{\mathrm{t}}^{\mathrm{e}}$ with an algebraic multiplicity of *two* (first and third rows in Table 1), the eigenvector solution is

	$\mathbf{p}_{\mathrm{t}} = \mathbf{0}$	$\mathbf{p}_{\mathrm{t}} \neq \mathbf{0}$
$\mathbf{q}_{\mathrm{t}} = \mathbf{0}$	Two eigenvectors, $\mathbf{n}\cdot\mathbf{w} = 0$	Two eigenvectors $\mathbf{n}\cdot\mathbf{w} = 0$
$\mathbf{q}_{\mathrm{t}} \neq \mathbf{0}$	Two eigenvectors $\mathbf{w} = w_{\mathrm{n}}\mathbf{n} + \alpha\bar{\mathbf{q}}_{\mathrm{t}} + \beta(\mathbf{n}\times\bar{\mathbf{q}}_{\mathrm{t}})$	One eigenvector $\mathbf{w} = \alpha\bar{\mathbf{p}}_{\mathrm{t}}$

(4.43)

where $\times$ denotes the vector cross product, α and β are arbitrary scalars, and, in the lower left solution,

$$w_n = \frac{\alpha}{\bar{\eta}} \frac{\bar{p}_n(\bar{\mathbf{q}}_t \cdot \bar{\mathbf{q}}_t)}{(1-\bar{\theta}_n)}. \tag{4.44}$$

If the Ottosen–Runesson conditions hold, only the upper left solution in (4.43) is possible.

If $x = x_t^e$ with an algebraic multiplicity of *unity* (last three rows in Table 1), the only eigenvector is

$$\boxed{\mathbf{w} = \alpha(\mathbf{n} \times \bar{\mathbf{q}}_t)}, \tag{4.45}$$

where α is an arbitrary scalar. This eigenvector will be perpendicular to the two remaining eigenvectors if $\mathbf{p}_t$ is coaxial with $\mathbf{q}_t$, as is the case with the specific flow law discussed in Section 4.6.

OTTOSEN and RUNESSON's (1991) solutions for the eigenvectors associated with x_t^e are incomplete. They implicitly assume (except, apparently, when the Ottosen–Runesson conditions hold) that the tangential projection of $\mathbf{p}$ is non-zero. Ottosen and Runesson state that the eigenvector associated with x_t^e must be perpendicular to both $\mathbf{n}$ and $\mathbf{q}$, which we have shown is correct only if the tangential part of $\mathbf{p}$ is non-zero (in Ottosen and Runesson, the notations for $\mathbf{p}$ and $\mathbf{q}$ are reversed and differ by a scalar multiplier).

Now consider the case that $x = x_n^e$. It is shown in the Appendix that when the algebraic multiplicity of x_n^e is *two* (fourth row in Table 1), the solution for the eigenvector $\mathbf{w}$ depends on the *normal* components of $\mathbf{p}$ and $\mathbf{q}$ as follows:

	$p_n = 0$	$p_n \neq 0$
$q_n = 0$	Two eigenvectors, $\mathbf{w} = \alpha\mathbf{n} + \beta\bar{\mathbf{p}}$	One eigenvector $\mathbf{w} = \alpha\mathbf{n}$
$q_n \neq 0$	One eigenvector $\mathbf{w} = \beta\bar{\mathbf{p}}$	

(4.46)

where α and β are arbitrary scalars. Any one of these solutions is possible even if the Ottosen–Runesson conditions hold.

When the algebraic multiplicity of x_n^e is *unity* (first and fifth rows in Table 1), the solution is

	$p_n = 0$	$p_n \neq 0$
$q_n = 0$	One eigenvector $\mathbf{w} = \alpha\mathbf{n}$	One eigenvector $\mathbf{w} = \alpha\mathbf{n}$
$q_n \neq 0$	One eigenvector $\mathbf{w} = w_n\mathbf{n} + \beta\bar{\mathbf{p}}$	

(4.47)

where α and β are arbitrary scalars, and, in the lower-left solution,

$$w_n = -\beta \frac{\bar{\eta}}{\bar{q}_n}(1+\bar{\theta}_t). \tag{4.48}$$

4.5. *Jump restriction implied by the spectral analysis*

Although the spectral analysis gives no information about the *magnitude* of the stress increment, it does imply a restriction on the evolution of stresses within the transition layer for a *given* wave speed, c^*. The stress and flow parameters must vary throughout the transition layer in such a way that the eigenvalue x^* satisfies the characteristic polynomial (4.25) *for every state* in the transition layer. If x^* is *not* equal to an elastic eigenvalue, then

$$\frac{\theta_n}{x_n^e - x^*} + \frac{\theta_t}{x_t^e - x^*} = 1 \quad \text{and} \quad \frac{[\![\theta_n]\!]}{x_n^e - x^*} + \frac{[\![\theta_t]\!]}{x_t^e - x^*} = 0 \tag{4.49}$$

throughout the shock; in other words, any shock propagating at a non-elastic wave speed will have a state path such that θ_n vs θ_t is a straight line tangent to the complex parabola in Fig. 2. If $x^* = x_n^e$, then θ_n must equal zero throughout the shock. In contrast, if $x^* = x_t^e$, the characteristic polynomial is automatically satisfied for all points within the shock, thereby leaving the state path unrestricted.

4.6. *Terms in the spectral analysis for a specific flow law*

The preceding spectral analysis holds for any physical interpretation for the second-order tensors **P** and **Q** (so long as they are independent of the strain rate). One of the better-known flow laws of the form (4.9) is that of RUDNICKI and RICE (1975), which is expressed in the following form by NEEDLEMAN and RICE (1978):

$$\mathbf{P} \equiv \mathbf{S}/2\tau_e + a\mathbf{I}/3, \tag{4.50a}$$

$$\mathbf{Q} \equiv \mathbf{S}/2\tau_e + b\mathbf{I}/3. \tag{4.50b}$$

Here, a and b are scalars, **S** is the deviatoric stress tensor, τ_e is the "equivalent" shear stress (i.e. $\tau_e \equiv \sqrt{\mathbf{S}:\mathbf{S}/2}$), and **I** is the identity tensor. The yield surface corresponding to (4.50b) is axisymmetric about the one-dimensional linear manifold of isotropic second-order tensors. In other words, the yield criterion is the same as the Mises criterion except that the flow stress is pressure dependent, as is commonly observed for rocks and porous metals. For this specific flow law, the tensor **Q** is parallel to the normal to the yield surface in stress space, and **P** is parallel to the plastic strain increment. Because **P** has an isotropic component, this material exhibits plastic compressibility.

Assuming elastic isotropy, (4.32) gives

$$\mathbf{p} = \frac{G}{\tau_e}\mathbf{n}\cdot\mathbf{S} + K\alpha\mathbf{n}, \tag{4.51a}$$

$$\mathbf{q} = \frac{G}{\tau_e}\mathbf{n}\cdot\mathbf{S} + Kb\mathbf{n}, \tag{4.51b}$$

and

$$\eta = h + G + Kab. \tag{4.52}$$

Typically, $ab \geqslant 0$ and $h > -G$, and, therefore, $\eta > 0$. The scalars θ_n and θ_t are

$$\theta_n = \frac{1}{\eta}\left(\frac{G}{\tau_e}S_{11} + Ka\right)\left(\frac{G}{\tau_e}S_{11} + Kb\right), \tag{4.53a}$$

$$\theta_t = \frac{1}{\eta}\frac{G^2}{\tau_e^2}[(S_{12})^2 + (S_{13})^2], \tag{4.53b}$$

where the components refer to an orthonormal basis with the 1-direction parallel to $\mathbf{n}$. With $\eta > 0$, the quantity θ_t is non-negative so that constitutive laws with (4.50) will always fall in regions **B** or **C** of Fig. 2, and, referring to Fig. 1, the state-dependent eigenvalues will be real, and no more than one can equal zero. Also note that θ_n is of first-order in $\frac{1}{2}(b+a)$, but second-order in $\frac{1}{2}(b-a)$; thus, plastic wave speeds for this material appear to be an order of magnitude more sensitive to plastic compressibility than to plastic non-normality.

Substituting (4.53) into (4.34) leads immediately to RUDNICKI and RICE's (1975) equation for the critical hardening modulus at localization. Because the model (4.50) is intended for porous materials, one might obtain a better prediction for the critical hardening modulus by using the *apparent* plastic flow direction $\mathbf{P}^*$ and the *apparent* hardening modulus h^* of (5.13).

5. Elastic–Plastic Coupling

In this section we demonstrate that dependence of the elastic moduli on plastic dilatation does not affect the fundamental structure of the non-associative tangent compliance tensor (4.9), so the results of the previous sections may be used for these materials.

If the elastic part of the strain response is linear but includes dependence of the elastic moduli on plastic straining, the elastic strain increment may be written as the sum of a recoverable part and a coupled part:

$$\mathrm{d}\boldsymbol{\varepsilon}^e = \mathbb{M} : \mathrm{d}\boldsymbol{\sigma} + \mathrm{d}\mathbb{M} : \boldsymbol{\sigma}, \tag{5.1}$$

where $\mathbb{M}$ is the instantaneous macroscopic fourth-order elastic compliance tensor.

Elastic–plastic coupling is commonly caused by the presence of microvoids, so we will discuss it in this context. ZHAO *et al.* (1989) give the macroscopic (effective) isotropic elastic moduli for a spherically voided material as

$$\frac{G}{G_m} = f_m\left(\frac{1-\beta_m}{1-f_m\beta_m}\right) \quad \text{where} \quad \beta_m = \frac{2}{15}\left(\frac{4-5\nu_m}{1-\nu_m}\right), \tag{5.2a}$$

$$\frac{K}{K_m} = f_m\left(\frac{1-\alpha_m}{1-f_m\alpha_m}\right) \quad \text{where} \quad \alpha_m = \frac{1}{3}\frac{1+\nu_m}{1-\nu_m}. \tag{5.2b}$$

Here, G and K are the shear and bulk moduli respectively, and f_m is the volume fraction of the matrix. A subscript "m" indicates the matrix material (e.g. G_m is the shear modulus of the matrix material). This model may be subsumed under a more general form for the macroscopic elastic compliance tensor, namely,

$$\mathbb{M} = \mathbb{M}_m + \frac{1-f_m}{f_m}\mathbb{M}_c, \tag{5.3}$$

where $\mathbb{M}_m$ is the elastic compliance tensor for the matrix material, and $\mathbb{M}_c$ is a fourth-order elastic–plastic coupling tensor dependent only on the matrix material properties. The tensor $\mathbb{M}_c$ corresponding to (5.2) is

$$\begin{aligned}\mathbb{M}_c &= \frac{1}{2G_m(1-\beta_m)}\mathbb{D}_s + \frac{1}{3K_m(1-\alpha_m)}\hat{\mathbf{I}}\hat{\mathbf{I}} \\ &= \frac{3(1-\nu_m)}{2G_m}\left[\frac{5}{7-5\nu_m}\mathbb{D}_s + \frac{1}{2(1+\nu_m)}\hat{\mathbf{I}}\hat{\mathbf{I}}\right],\end{aligned} \tag{5.4}$$

where $\mathbb{D}_s$ is the symmetric deviator operator of equation (3.27), $\hat{\mathbf{I}}$ is the identity tensor divided by its Euclidean magnitude, and $\hat{\mathbf{I}}\hat{\mathbf{I}}$ is a fourth-order tensor-tensor dyad having components

$$(\hat{\mathbf{I}}\hat{\mathbf{I}})_{ijrs} = \tfrac{1}{3}\delta_{ij}\delta_{rs}. \tag{5.5}$$

Assuming the spherical voids remain spherical after permanent deformation [which, according to RICE and TRACEY (1969), is reasonable for high ratios of pressure to effective stress], and assuming the elastic moduli of the matrix material are independent of plastic straining, the increment of the macroscopic elastic compliance tensor in (5.3) is

$$\mathrm{d}\mathbb{M} = -\frac{\mathrm{d}f_m}{f_m^2}\mathbb{M}_c. \tag{5.6}$$

For a plastically incompressible matrix material, the rate of change of void volume fraction f_v at the *unloaded* state depends on the rate of plastic deformation according to the *kinematic* relationship

$$\dot{f}_v = (1-f_v)\,\mathrm{tr}\,\mathbf{D}^p, \tag{5.7}$$

where $\mathbf{D}^p$ is the plastic part of the rate of deformation and may be approximated by the plastic strain rate for small displacement gradient deformations. Equation (5.7) is used by several authors (e.g. NEEDLEMAN and RICE, 1978), except that the distinction between the current void fraction and the void fraction at the unloaded state is ignored in these works. It is straightforward to show that the difference is negligible if elastic dilatation is negligible compared to plastic dilatation. However, even if the elastic *deformation* is small compared to the plastic deformation, the elastic dilatation may still be large compared to the plastic dilatation—after all, for non-voided materials,

the plastic dilatation is identically zero and so the elastic dilatation is indeed large in comparison. Equation (5.7) is a kinematic equation—it does not depend on the mechanism of void growth. However, the void rate in (5.7) is frequency interpreted in the literature as the contribution resulting from void *growth* only, with other mechanisms such as void nucleation contributing separately. This approach is legitimate if f_v is regarded as an *effective* void fraction; for example, the effective void fraction might be defined as the actual void fraction plus the volume fraction of cracked or debonded particles. Even if this approach is adopted, however, the effective void fraction should be carefully distinguished from the true void fraction when writing equivalent work expressions equating macroscopic plastic work rate to the matrix plastic work rate because the *true* matrix fraction (not $1-f_{v,\text{effective}}$) should be used in these expressions.

In the present analysis, f_v is the true unloaded void fraction so that $f_v+f_m=1$. Equation (5.7) may therefore be integrated to give

$$f_m = f_m^o/J_p, \tag{5.8}$$

where f_m^o is the initial matrix fraction, and J_p is the accumulated plastic Jacobian (equal to the ratio of the initial unloaded macroscopic density to the current unloaded macroscopic density). The rate of macroscopic elastic compliance may now be written

$$\dot{\mathbb{M}} = \frac{\dot{J}_p}{f_m^o}\mathbb{M}_c, \tag{5.9}$$

which, for small displacement gradients, becomes, in incremental form,

$$d\mathbb{M} = \left(\frac{1}{f_m^o}\mathbb{M}_c\right)\text{tr}\,(d\boldsymbol{\varepsilon}^p). \tag{5.10}$$

Thus, using (3.1) and (5.1) together with a plastic strain increment of the form

$$d\boldsymbol{\varepsilon}^p = \frac{1}{h}\mathbf{PQ}:d\boldsymbol{\sigma}, \tag{5.11}$$

the tangent compliance tensor ξ^{-1} of (4.2) becomes

$$\xi^{-1} = \mathbb{M} + \frac{1}{h^*}\mathbf{P}^*\mathbf{Q}, \tag{5.12}$$

where

$$\frac{1}{h^*}\mathbf{P}^* = \frac{1}{h}\mathbf{P} + \frac{1}{hf_m^o}(\mathbf{I}:\mathbf{P})\mathbb{M}_c:\boldsymbol{\sigma}. \tag{5.13}$$

Neglecting elastic–plastic coupling, HILL (1968) proved that normality of the plastic strain increment to the yield surface implies self-adjointness of the constitutive tensor ξ. Equation (5.12) shows that even if the plastic strain rate is normal to the yield surface (i.e. if $\mathbf{P}$ is coaxial with $\mathbf{Q}$), elastic–plastic coupling generally makes the tangent compliance tensor ξ^{-1} non-self-adjoint. Non-self-adjointness of associative flow laws

has also been noted by MAIER and HUECKEL (1979) for elastic–plastic coupling and by TVERGAARD (1982) for thermo-mechanical coupling.

For isotropic elastic response, if the deviatoric part of **P** is coaxial with the stress deviator, then the deviatoric part of **P*** is also coaxial with the stress deviator. If, as in the specific flow law of Section 4.6, the deviatoric parts of *both* **P** and **Q** are coaxial with the stress deviator, then [assuming η in (4.13) is non-negative] elastic–plastic coupling will *not* produce complex eigenvalues for shock waves.

6. DISCUSSION

The first analysis presented in this paper serves to bound the *magnitude* of a proposed jump in stress. With our example, we demonstrated that in some cases the magnitude of the jump in stress can be shown to be zero, thereby ruling out the existence of such a jump altogether.

The extended spectral analysis gives no information about the magnitude of a proposed jump in stress. Instead, we used the eigen-analysis to derive explicit solutions for the permissible speeds at which a jump can travel and the *direction* for the jump in stress.

These results can be used together to restrict the class of possible solutions to boundary value problems, such as dynamic crack growth, that may involve moving stress discontinuities.

ACKNOWLEDGEMENTS

Support of this work by the Mechanics Division of the Office of Naval Research under Grants N00014-89-J-1206 and N00014-92-J-1314, and by the Mechanics and Materials Program of the National Science Foundation under Grant MSS-8552486 is gratefully acknowledged. We also wish to thank an anonymous reviewer for very detailed, helpful comments.

REFERENCES

ASARO, R. J. 1979 *Acta Metall.* **27,** 445.

ASARO, R. J. 1983 In *Advances in Applied Mechanics*, Vol. 23, p. 1. Academic Press, New York.

BECKER, R. 1987 *J. Mech. Phys. Solids* **35,** 577.

BEREZIN, A. V. 1987 *Mech. Solids* **22,** 88.

BRIDGMAN, P. W. 1952 *Studies in Large Plastic Flow and Fracture*. McGraw-Hill, New York.

CHADWICK, P. 1976 *Continuum Mechanics: Concise Theory and Problems*. Wiley, New York.

CHANDLER, H. W. 1985 *J. Mech. Phys. Solids* **33,** 215.

COURANT, R. and FRIEDRICHS, K. O. 1948 *Supersonic Flow and Shock Waves*. (Third Printing: 1985). Springer, Berlin.

DRUGAN, W. J. 1986 *J. appl. Mech.* **53,** 224.

DRUGAN, W. J. and RICE, J. R. 1984 In *Mechanics of Material Behavior* (edited by G. J. DVORAK and R. T. SHIELD), p. 59. Elsevier Science, Amsterdam.

DRUGAN, W. J. and SHEN, Y.	1987	*J. Mech. Phys. Solids* **35,** 771.
DRUGAN, W. J. and SHEN, Y.	1990	*J. Mech. Phys. Solids* **38,** 553.
GURSON, A. L.	1977	*J. Engng Mater. Technol.* **99,** 2.
HADAMARD, J.	1903	*Lecons sur la Propagation des Ondes et les Equations de l'Hydrodynamique*. Hermann, Paris.
HILL, R.	1952	*J. Mech. Phys. Solids* **1,** 19.
HILL, R.	1961	In *Progress in Solid Mechanics* (edited by R. Hill and I. N. Sneddon), Vol. 2, p. 244. North Holland, Amsterdam.
HILL, R.	1962	*J. Mech. Phys. Solids* **10,** 1.
HILL, R.	1967	*J. Mech. Phys. Solids* **15,** 79.
HILL, R.	1968	*J. Mech. Phys. Solids* **16,** 315.
HILL, R. and HUTCHINSON, J. W.	1975	*J. Mech. Phys. Solids* **23,** 239.
HORI, M. and NEMAT-NASSER, S.	1989	In *Advances in Fracture Research*, p. 333. Proc. 7th International Conference on Fracture, c. 1989. Pergamon, Oxford.
LEIGHTON, J. T., CHAMPION, C. R. and FREUND, L. B.	1987	*J. Mech. Phys. Solids* **35,** 541.
LORET, B. and HARIRECHE, O.	1991	*J. Mech. Phys. Solids* **39,** 569.
LORET, B., PREVOST, J. H. and HARIRECHE, O.	1990	*Eur. J. Mech. A Solids* **9,** 225.
MAIER, G. and HUECKEL, T.	1979	*Int. J. Rock Mech. Min. Sci. Geomech. Abstr.* **16,** 77.
MANDEL, J.	1962	*J. Méch.* **1,** 3.
MANDEL, J.	1963	In *Stress Waves in Anelastic Solids* (edited by H. KOLSKY and W. PRAGER), p. 331. IUTAM Symposium, Providence, c. 1964. Springer, Berlin.
MANDEL, J.	1964	In *Rheology and Soil Mechanics* (edited by J. KRAVTCHENKO and P. M. SIRIEYS), p. 58. International Symposium, Grenoble, c. 1966. Springer, New York.
MRÓZ, Z. and RANIECKI, B.	1976	*Int. J. Engng Sci.* **14,** 211.
NEEDLEMAN, A. and RICE, J. R.	1978	In *Mechanics of Sheet Metal Forming* (edited by D. P. KOISTINEN and N.-M. WANG), p. 237. Plenum, New York.
NEMAT-NASSER, S. and OBATA, M.	1990	*Int. J. Fracture* **42,** 287.
NIKOLIC, R. R. and RICE, J. R.	1988	*Mech. Mater.* **7,** 163.
OHNO, N. and HUTCHINSON, J. W.	1984	*J. Mech. Phys. Solids* **32,** 63.
OTTOSEN, N. S. and RUNESSON, K.	1991	*Int. J. Solids Struct.* **28,** 135.
PASTOR, M., ZIENKIEWICZ, O. C. and CHAN, A. H. C.	1990	*Int. J. numer. analyt. Meth. Geomech.* **14,** 151.
RANIECKI, R. and BRUHNS, O. T.	1981	*J. Mech. Phys. Solids* **29,** 153.
RICE, J. R.	1976	In *Theoretical and Applied Mechanics* (edited by W. T. KOITER), p. 207. North-Holland, Amsterdam.
RICE, J. R. and TRACEY, D. M.	1969	*J. Mech. Phys. Solids* **17,** 201.
RUDNICKI, J. W. and RICE, J. R.	1975	*J. Mech. Phys. Solids* **23,** 371.
SHEN, Y. and DRUGAN, W. J.	1990	*J. appl. Mech.* **57,** 569.
SPITZIG, W. A. and RICHMOND, O.	1984	*Acta Metall.* **32,** 457.
SPITZIG, W. A., SOBER, R. J. and RICHMOND, O.	1975	*Acta Metall.* **23,** 885.
STOREN, S. and RICE, J. R.	1975	*J. Mech. Phys. Solids* **23,** 421.
TING, T. C. T.	1976	In *ASME AMD* (edited by E. VARLEY), Vol. 17, p. 41.

TVERGAARD, V. 1982 *J. Mech. Phys. Solids* **30,** 399.
TVERGAARD, V. 1990 In *Advances in Applied Mechanics*, Vol. 27, p. 83. Academic Press, New York.
TVERGAARD, V., NEEDLEMAN, A. and LO, K. K. 1981 *J. Mech. Phys. Solids* **29,** 115.
WALLACE, D. C. 1980 *Phys. Rev.* **B22,** 1477, 1487 and 1495 (three articles).
ZHAO, Y. H., TANDON, G. P. and WENG, G. J. 1989 *Acta Mechanica* **76,** 105.

APPENDIX: DERIVATION OF THE EIGENVECTORS ASSOCIATED WITH ELASTIC EIGENVALUES

1. *Solution when* $x = x_t^e$

With $\bar{x} = \bar{x}_t^e$, the system (4.40) becomes

$$\bar{\mathbf{p}}_t(\bar{q}_n w_n + \bar{\mathbf{q}}_t \cdot \mathbf{w}_t) = \mathbf{0}, \tag{A.1a}$$

$$-w_n + \frac{1}{\bar{\eta}} \bar{p}_n(\bar{q}_n w_n + \bar{\mathbf{q}}_t \cdot \mathbf{w}_t) = 0, \tag{A.1b}$$

where we have used (4.28) to write $\bar{x}_t^e - \bar{x}_n^e = -1$.

First consider the case that the algebraic multiplicity of x_t^e is two or three, which, by (4.36), is possible if and only if $\theta_t = 0$. Thus, recalling (4.23b), exactly one of the following holds.

Case 1: $\mathbf{p}_t = \mathbf{0}$ and $\mathbf{q}_t = \mathbf{0}$. Here, (A.1a) is automatically satisfied and, using (4.23a), (A.1b) becomes

$$(\bar{\theta}_n - 1)w_n = 0. \tag{A.2}$$

If the algebraic multiplicity of x_t^e is three, (4.36) shows that $\bar{\theta}_n = 1$, and the eigenvectors are, therefore, arbitrary. If the algebraic multiplicity of x_t^e is two, (4.36) shows that $\bar{\theta}_n \neq 1$, and, therefore, the eigenvectors must satisfy $\mathbf{n} \cdot \mathbf{w} = 0$, but are otherwise arbitrary.

Case 2: $\mathbf{p}_t \neq \mathbf{0}$ and $\mathbf{q}_t = \mathbf{0}$. Equation (A.1) becomes

$$\bar{q}_n w_n = 0, \tag{A.3a}$$

$$-w_n + \frac{1}{\bar{\eta}} \bar{p}_n(\bar{q}_n w_n) = 0, \tag{A.3b}$$

which show that $\mathbf{n} \cdot \mathbf{w} = 0$.

Case 3: $\mathbf{p}_t = \mathbf{0}$ and $\mathbf{q}_t \neq \mathbf{0}$. Equation (A.1a) is automatically satisfied and, recalling (4.23a), (A.1b) becomes

$$(\bar{\theta}_n - 1)w_n + \frac{1}{\bar{\eta}} \bar{p}_n(\bar{\mathbf{q}}_t \cdot \mathbf{w}_t) = 0. \tag{A.4}$$

If the algebraic multiplicity of x_t^e is *three* (i.e. if $\bar{\theta}_n = 1$), then the only restriction on the eigenvector is $\bar{\mathbf{q}}_t \cdot \mathbf{w} = 0$, which follows from (A.4) because $\bar{\theta}_n = 1$ guarantees $p_n \neq 0$. If, on the other hand, the algebraic multiplicity of x_t^e is *two* (i.e. if $\bar{\theta}_n \neq 1$), then (A.4) is a restriction between the normal component of $\mathbf{w}$ and its component in the direction of $\bar{\mathbf{q}}_t$. By the projection theorem, there must exist scalars w_n, α and β, such that

$$\mathbf{w} = w_n \mathbf{n} + \alpha \bar{\mathbf{q}}_t + \beta(\mathbf{n} \times \bar{\mathbf{q}}_t), \tag{A.5}$$

where $\times$ is the vector cross product. Equation (A.4) shows that

$$w_n = \frac{\alpha}{\bar{\eta}} \frac{\bar{p}_n(\bar{\mathbf{q}}_t \cdot \bar{\mathbf{q}}_t)}{(1 - \bar{\theta}_n)}, \tag{A.6}$$

and the scalars α and β are unrestricted.

Case 4: $\mathbf{p}_t \neq \mathbf{0}$ and $\mathbf{q}_t \neq \mathbf{0}$, with $\mathbf{p}_t$ perpendicular to $\mathbf{q}_t$. Equation (A.1) becomes

$$(\bar{q}_n w_n + \bar{\mathbf{q}}_t \cdot \mathbf{w}_t) = 0, \tag{A.7a}$$

$$-w_n + \frac{1}{\bar{\eta}} \bar{p}_n (\bar{q}_n w_n + \bar{\mathbf{q}}_t \cdot \mathbf{w}_t) = 0. \tag{A.7b}$$

Using (A.7a) in (A.7b) shows that $w_n = 0$, which, from (A.7a), implies $\bar{\mathbf{q}}_t \cdot \mathbf{w} = 0$. Thus, the eigenvector $\mathbf{w}$ must be perpendicular to both $\mathbf{n}$ and $\bar{\mathbf{q}}_t$, or, in other words,

$$\mathbf{w} = \alpha(\mathbf{n} \times \bar{\mathbf{q}}_t), \tag{A.8a}$$

where α is an arbitrary scalar. Because $\mathbf{p}_t$ is perpendicular to $\mathbf{q}_t$ and (by definition) to $\mathbf{n}$, the eigenvector may alternatively be written

$$\mathbf{w} = \alpha \bar{\mathbf{p}}_t, \tag{A.8b}$$

which concludes the solution for the eigenvectors associated with x_t^e having an algebraic multiplicity of two or three.

If the algebraic multiplicity of x_t^e is *unity*, (4.36) shows that $\bar{\theta}_t \neq 0$. Thus, from (4.23b), $\mathbf{p}_t$ and $\mathbf{q}_t$ must be non-zero and non-perpendicular. The system (A.7) still applies, and the eigenvector is given by (A.8a).

2. *Solution when* $x = x_n^e$

With $\bar{x} = \bar{x}_n^e$, the decomposed eigen-system (4.40) becomes

$$\mathbf{w}_t + \frac{1}{\bar{\eta}} \bar{\mathbf{p}}_t (\bar{q}_n w_n + \bar{\mathbf{q}}_t \cdot \mathbf{w}_t) = \mathbf{0}, \tag{A.9a}$$

$$\bar{p}_n (\bar{q}_n w_n + \bar{\mathbf{q}}_t \cdot \mathbf{w}_t) = 0, \tag{A.9b}$$

where we have used (4.28) to write $\bar{x}_n^e - \bar{x}_t^e = 1$. By (4.35), the eigenvalue x can equal x_n^e if and only if $\theta_n = 0$. Thus, recalling (4.23a), exactly one of the following holds.

Case 1: $p_n = 0$ and $q_n = 0$. Here, (A.9b) is automatically satisfied and (A.9a) becomes

$$\mathbf{w}_t + \frac{1}{\bar{\eta}} \bar{\mathbf{p}}_t (\bar{\mathbf{q}}_t \cdot \mathbf{w}_t) = \mathbf{0}, \tag{A.10}$$

which shows that $\mathbf{w}_t = \beta \bar{\mathbf{p}}_t$, where β is a scalar. Using (4.23b), (A.10) requires that β satisfy

$$(1 + \bar{\theta}_t)\beta \bar{\mathbf{p}}_t = \mathbf{0}. \tag{A.11}$$

If the algebraic multiplicity of x_n^e is *two*, (4.35) shows that $\bar{\theta}_t = -1$, and, therefore, $\mathbf{w} = \alpha \mathbf{n} + \beta \bar{\mathbf{p}}_t$, where α and β are arbitrary scalars. If, on the other hand, the algebraic multiplicity of x_n^e is unity, (4.35) shows that $\bar{\theta}_t \neq -1$, and, therefore, $\beta \bar{\mathbf{p}}_t = 0$ and the eigenvector is $\mathbf{n}$.

Case 2: $p_n \neq 0$ and $q_n = 0$. The system (A.9) becomes

$$\mathbf{w}_t + \frac{1}{\bar{\eta}} \bar{\mathbf{p}}_t (\bar{\mathbf{q}}_t \cdot \mathbf{w}_t) = \mathbf{0}, \tag{A.12a}$$

$$\bar{\mathbf{q}}_t \cdot \mathbf{w}_t = 0, \tag{A.12b}$$

which implies that $\mathbf{w}_t = \mathbf{0}$, or, in other words, $\mathbf{w} = \mathbf{n}$.

Case 3: $p_n = 0$ and $q_n \neq 0$. Equation (A.9b) is automatically satisfied and (A.9a) becomes

$$\mathbf{w}_t + \frac{1}{\bar{\eta}} \bar{\mathbf{p}}_t (\bar{q}_n w_n + \bar{\mathbf{q}}_t \cdot \mathbf{w}_t) = \mathbf{0}, \tag{A.13}$$

which, because $p_n = 0$, shows that $\mathbf{w} = w_n \mathbf{n} + \beta \bar{\mathbf{p}}$. Equation (A.13) requires

$$\tilde{\mathbf{p}}_t\left[(1+\tilde{\theta}_t)\beta+\frac{1}{\tilde{\eta}}\tilde{q}_n w_n\right]=\mathbf{0}. \tag{A.14}$$

Thus,

$$\mathbf{w}=\left[-\beta\frac{\tilde{\eta}}{\tilde{q}_n}(1+\tilde{\theta}_t)\right]\mathbf{n}+\beta\tilde{\mathbf{p}}. \tag{A.15}$$

If the algebraic multiplicity of x_n^e is *two* (i.e. if $\tilde{\theta}_t=-1$), the eigenvector becomes simply $\beta\tilde{\mathbf{p}}$, where β is arbitrary. If $\tilde{\mathbf{p}}_t=\mathbf{0}$ (in which case $\mathbf{p}=\mathbf{0}$, and, therefore, $\theta_n=\theta_t=0$), the eigenvector reduces to the elastic eigenvector $\alpha\mathbf{n}$, where α is arbitrary.