Outline of the derivation of Cauchy Equations of Motion

Euler's 1st and 2nd laws These integral equations apply to any sub-body B^*:

Net force = rate of linear momentum: $$\underset{\sim}{\mathbf{F}}^{\mathrm{NET}} = \frac{D}{Dt}\int_{B^*} \underset{\sim}{\mathbf{v}}\rho\, dV. \tag{1}$$

Net torque = rate of angular momentum: $$\underset{\sim}{\mathbf{T}}^{\mathrm{NET}} = \frac{D}{Dt}\int_{B^*} \underset{\sim}{\mathbf{x}} \times \underset{\sim}{\mathbf{v}}\rho\, dV. \tag{2}$$

Utilize conservation of mass Noting that $\rho dV = \rho_o dV_o$, conservation of mass permits the time derivative to be brought inside the integrals, so that

$$\underset{\sim}{\mathbf{F}}^{\mathrm{NET}} = \int_{B^*} \underset{\sim}{\mathbf{a}}\rho\, dV. \qquad \text{(Generalizes Newton's 2nd law, F=ma)} \tag{3}$$

$$\underset{\sim}{\mathbf{T}}^{\mathrm{NET}} = \int_{B^*} \underset{\sim}{\mathbf{x}} \times \underset{\sim}{\mathbf{a}}\rho\, dV, \tag{4}$$

where $\underset{\sim}{\mathbf{a}} = \dot{\underset{\sim}{\mathbf{v}}}$, and Eq. (4) also employed the fact that $\dot{\underset{\sim}{\mathbf{x}}} = \underset{\sim}{\mathbf{v}}$ so that $\dot{\underset{\sim}{\mathbf{x}}} \times \underset{\sim}{\mathbf{v}} = \underset{\sim}{\mathbf{0}}$.

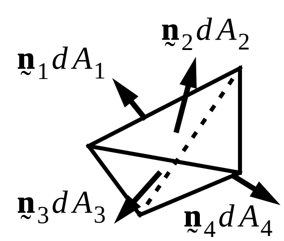
Stress principle of Euler and Cauchy: The effect of an external body can be represented through a traction (force per area) vector $\underset{\sim}{\mathbf{t}}$. Then, letting $\underset{\sim}{\mathbf{b}}$ denote the body force per unit mass (e.g. gravity), and assuming no point forces, no point moments, and no distributed moments (e.g. no dipoles in an electric field), the net force and net torque may be written:

$$\underset{\sim}{\mathbf{F}}^{\mathrm{NET}} = \int_{\partial B^*} \underset{\sim}{\mathbf{t}}\, dS + \int_{B^*} \underset{\sim}{\mathbf{b}}\rho\, dV \tag{5}$$

$$\underset{\sim}{\mathbf{T}}^{\mathrm{NET}} = \int_{\partial B^*} \underset{\sim}{\mathbf{x}} \times \underset{\sim}{\mathbf{t}}\, dS + \int_{B^*} \underset{\sim}{\mathbf{x}} \times \underset{\sim}{\mathbf{b}}\rho\, dV \tag{6}$$

Cauchy's fundamental stress theorem Begin with a key assumption that, in addition to varying in space and time, the traction $\underset{\sim}{\mathbf{t}}$ is also a function of the unit normal $\underset{\sim}{\mathbf{n}}$ of the surface. Cauchy's fundamental theorem states that this dependence is *linear* and consequently there exists a tensor $\underset{\approx}{\sigma}(\underset{\sim}{\mathbf{x}}, t)$ such that $\underset{\sim}{\mathbf{t}} = \underset{\approx}{\sigma} \bullet \underset{\sim}{\mathbf{n}}$. Outline of proof:

i. Start with a small tetrahedron with sides labeled 1 through 4.



ii. Put $\underset{\sim}{\mathbf{t}} = \underset{\sim}{\mathbf{t}}(\underset{\sim}{\mathbf{x}}, t, \underset{\sim}{\mathbf{n}})$ in Eq. (5), and this into Euler's 1st law, Eq. (1). As the size of the tetrahedron goes to zero, the surface integral dominates (it is of order L^2 with respect to the tetrahedron's characteristic length L, whereas volume integrals are of order L^3). As the tetrahedron shrinks to a point, Euler's 1st law requires that the vector sums of the forces (traction times area) on the faces must equilibrate:

$$\sum_{k=1}^{4} \underset{\sim}{\mathbf{t}}(\underset{\sim}{\mathbf{n}}_k)A_k = \underset{\sim}{\mathbf{0}}, \text{ and therefore } \underset{\sim}{\mathbf{t}}(\underset{\sim}{\mathbf{n}}_4) = \sum_{k=1}^{3} \alpha_k \underset{\sim}{\mathbf{t}}(\underset{\sim}{\mathbf{n}}_k) \text{ where } \alpha_k = -\frac{A_k}{A_4}. \tag{7}$$

iii. For any body B, $\int_B \underset{\sim}{\mathbf{n}}\, dA = \underset{\sim}{\mathbf{0}}$. Applying this fact to the tetrahedron gives

$$\underset{\sim}{\mathbf{n}}_4 = \sum_{k=1}^{3} \alpha_k \underset{\sim}{\mathbf{n}}_k \tag{8}$$

iv. Putting (8) into (7) shows that $\underset{\sim}{\mathbf{t}}\left(\sum_{k=1}^{3} \alpha_k \underset{\sim}{\mathbf{n}}_k\right) = \sum_{k=1}^{3} \alpha_k \underset{\sim}{\mathbf{t}}(\underset{\sim}{\mathbf{n}}_k)$, which holds for any proportions of the tetrahedron, and therefore the dependence on the unit normal must be linear[1], completing the proof of existence of $\underset{\approx}{\sigma}$.

Cauchy's first law. Putting $\underset{\sim}{\mathbf{t}} = \underset{\approx}{\sigma} \bullet \underset{\sim}{\mathbf{n}}$ into (5) and applying the divergence theorem, Euler's 1st law becomes:

$$\int_{B^*} \underset{\approx}{\sigma} \bullet \overleftarrow{\nabla} dV + \int_{B^*} \underset{\sim}{\mathbf{b}} \rho dV = \int_{B^*} \underset{\sim}{\mathbf{a}} \rho dV \tag{9}$$

This must hold for all B^*, so the integrands must be equal. This gives the local form of balance of linear momentum, AKA "Cauchy's 1st law":

$$\boxed{\underset{\approx}{\sigma} \bullet \overleftarrow{\nabla} + \underset{\sim}{\mathbf{b}} \rho = \underset{\sim}{\mathbf{a}} \rho} \tag{10}$$

Cauchy's second law. Similarly putting $\underset{\sim}{\mathbf{t}} = \underset{\approx}{\sigma} \bullet \underset{\sim}{\mathbf{n}}$ into Eq. (6), applying the divergence theorem leads to the opportunity to apply the product rule $(\underset{\sim}{\mathbf{x}} \times \underset{\approx}{\sigma}) \bullet \overleftarrow{\nabla} = \underset{\sim}{\mathbf{x}} \times (\underset{\approx}{\sigma} \bullet \overleftarrow{\nabla}) - \underset{\approx}{\varepsilon}{:}\underset{\approx}{\sigma}$, Then Euler's 2nd law becomes

$$\int_{B^*} \underset{\approx}{\varepsilon}{:}\underset{\approx}{\sigma} dV = \int_{B^*} [\underset{\sim}{\mathbf{x}} \times (\underset{\approx}{\sigma} \bullet \overleftarrow{\nabla} + \underset{\sim}{\mathbf{b}} \rho - \underset{\sim}{\mathbf{a}} \rho)] dV \tag{11}$$

The right hand side is zero due to Cauchy's 1st law. This must hold for all B^* and therefore the left integrand $\underset{\approx}{\varepsilon}{:}\underset{\approx}{\sigma}$ must be zero, which is possible only if $\underset{\approx}{\sigma}$ is symmetric. This simple conclusion is the local form of balance of angular momentum, AKA Cauchy's 2nd law:

$$\boxed{\underset{\approx}{\sigma} = \underset{\approx}{\sigma}^T} \quad \text{Beware: this would not be true if there were distributed moments!} \tag{12}$$

Incidentally, this result allows us to write $\underset{\approx}{\sigma} \bullet \overleftarrow{\nabla}$ as $\overrightarrow{\nabla} \bullet \underset{\approx}{\sigma}$, which will probably be more familiar to your readers.

1. By positivity of area, the α_k coefficients are always negative, so it might seem that linearity has been proved only for negative coefficients. However, by applying Euler's 1st law to a thin wafer of material in the limit as the wafer's thickness goes to zero, we obtain Newton's 3rd law that every action has an equal and opposite reaction; that is, $\underset{\sim}{\mathbf{t}}(-\underset{\sim}{\mathbf{n}}_k) = -\underset{\sim}{\mathbf{t}}(\underset{\sim}{\mathbf{n}}_k)$.

BOUNDARY VALUE PROBLEMS

TABLE 1. Governing equations (independent variables are time and position)

Description	Equation	number of independent equations	Unknown dependent variables
Conservation of mass (AKA continuity)	$\dot{\rho} = -\rho\vec{\nabla} \bullet \underset{\sim}{\mathbf{v}}$	1	single scalar ρ, three v_k
Balance of linear momentum	$\rho\underset{\sim}{\mathbf{a}} = \vec{\nabla} \bullet \underset{\approx}{\sigma} + \rho\underset{\sim}{\mathbf{b}}$	3	three a_k, nine σ_{ij}
Balance of angular momentum	$\underset{\approx}{\sigma}^T = \underset{\approx}{\sigma}$	3	
Kinematical relation	$\underset{\sim}{\mathbf{v}} = \dot{\underset{\sim}{\mathbf{u}}}$	3	three u_k
Kinematical relation	$\underset{\sim}{\mathbf{a}} = \dot{\underset{\sim}{\mathbf{v}}}$	3	
Strain-displacement relation	$\underset{\approx}{\mathbf{e}} = \frac{1}{2}(\underset{\approx}{\mathbf{h}} + \underset{\approx}{\mathbf{h}}^T - \underset{\approx}{\mathbf{h}}^T \bullet \underset{\approx}{\mathbf{h}})$, where $h_{ij} = \left(\frac{\partial u_i}{\partial x_j}\right)_t$.	6	six e_{ij}
Constitutive law. Varies depending on the material. Generally relates stress to strain gradients and/or velocity gradients.	Elasticity: $\underset{\approx}{\sigma} = f(\underset{\approx}{\mathbf{e}})$, or Viscous: $\underset{\approx}{\sigma} = f(\underset{\approx}{\mathbf{D}})$, where $D_{ij} = \frac{1}{2}\left[\left(\frac{\partial v_i}{\partial x_j}\right)_t + \left(\frac{\partial v_j}{\partial x_i}\right)_t\right]$	6	

There are 25 equations and 25 unknowns. Constitutive equations are necessary to close the system of equations. The body force is presumed known (if unknown, it may be related to other fields which are governed by further equations. For example, an electrical repulsive force is dictated by the response of a charge density to an electric field, which is in turn governed by Maxwell's equations). Above, the time rates are material rates — they hold the reference position $\underset{\sim}{\mathbf{X}}$ constant. However the gradients in the above equations are with respect to the spatial position $\underset{\sim}{\mathbf{x}}$. To convert the above system to only two independent variables, $\underset{\sim}{\mathbf{x}}$ and t, the material derivatives must be converted (e.g., $\dot{\rho} = (\partial\rho/\partial t)_{\underset{\sim}{\mathbf{x}}} + \underset{\sim}{\mathbf{v}} \bullet \vec{\nabla}\rho$). After making these substitutions, it becomes clear that these equations are highly nonlinear and must be solved numerically (except for highly simplified geometries or constitutive equations). For such boundary value problems, initial values for the dependent variables are usually be specified. The spatial boundary conditions can be expressed in several possible forms:

1. Prescribed displacement: $\underset{\sim}{\mathbf{u}} = \underset{\sim}{\mathbf{u}}^*(\underset{\sim}{\mathbf{x}}, t)$ on ∂B_u. For example, a fixed end has zero displacement.
2. Prescribed traction: $\underset{\sim}{\mathbf{t}} = \underset{\sim}{\mathbf{t}}^*(\underset{\sim}{\mathbf{x}}, t)$ on ∂B_t. For example, a free surface has zero traction.
3. Single components: On a surface you can have one traction component specified and a different component of displacement specified. For example, a frictionless surface has zero traction components in the plane and zero displacement components normal to the plane.
4. Most generally, B.C.s are of the form $f(\underset{\sim}{\mathbf{t}}, \underset{\sim}{\mathbf{u}}) = 0$. For example, a spring boundary condition would make traction proportional to displacement.