



# ON THE DISTINCTION BETWEEN LARGE DEFORMATION AND LARGE DISTORTION FOR ANISOTROPIC MATERIALS

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Reference: Brannon, R.M., *Acta Mech*, Vol. 129, 1998.

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## INTRODUCTION: Is there a Best Strain?

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Seth-Hill family of strain measures:  $\varepsilon = \frac{1}{\kappa} \left[ \left( \frac{I}{I_0} \right)^\kappa - 1 \right]$  or  $\varepsilon \approx \frac{1}{\kappa} [ \mathbf{V}^\kappa - \mathbf{I} ]$

Lagrange  
 $\kappa=2$

Engineering  
 $\kappa=1$

“true”  
 $\kappa = -1$

Logarithmic  
 $\kappa=0$

If stress is a function of any of these, it can be written as a function of any of the others, so the idea of a “best strain” is problematic.

### KEY QUESTION:

What is the precise *mathematically testable* criterion by which one strain measure is declared better than another?

The **best strain** is perceived to be the one that makes the stress-strain function most linear. This seems to be a poor criterion since different materials would have different “best strain measures.”



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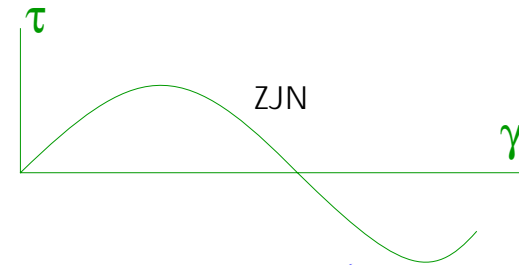
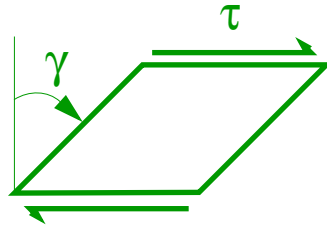
# Best objective rate?

Well-known result (Dienes, 1979)

ZJN  
Rate of stress

"Rate" of  
Deformation

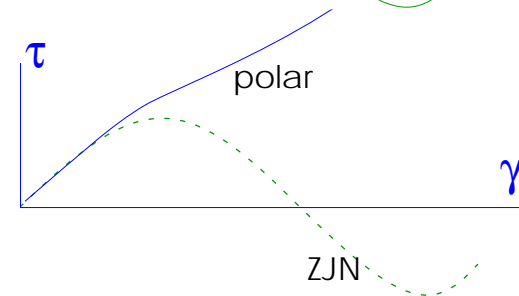
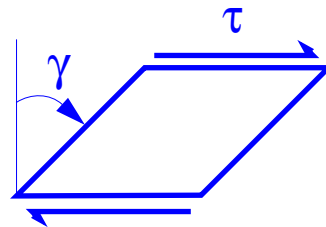
$$\dot{\sigma}_{ij}^* = \xi_{ijkl} D_{kl}$$



POLAR  
Rate of stress

Rate of  
Deformation

$$\dot{\sigma}_{ij}^o = \xi_{ijkl} D_{kl}$$



Usual conclusion: ZJN rate is "bad."

Alternative conclusion:

**Stress-strain relation must be *nonlinear*.**



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## If polar stress rate is linear in $\mathbf{D}$ , then so is the Jaumann stress rate.

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Dienes' counterexample assumed the objective stress “rate” to equal  $\xi_{ijkl}D_{kl}$ , where the stiffness  $\xi_{ijkl}$  was *constant*. That's a *linear* stress-strain law. Who says it should be linear?

**Claim:** The pathological behavior of the Jaumann rate cannot be blamed on the Jaumann rate per se — *linearity* is the culprit!

**Proof:** Suppose that  $\overset{\circ}{\sigma}_{ij} = \xi_{ijkl}D_{kl}$  is deemed to be a “good” constitutive law. The difference  $\overset{\circ}{\sigma}_{ij} - \overset{*}{\sigma}_{ij}$  between the polar and Jaumann rates is itself linear in  $\mathbf{D}$ . Thus, if the “good” equation is true, then there *does* exist a fourth-order tensor  $\mathbf{L}$  such that  $\overset{*}{\sigma}_{ij} = L_{ijkl}D_{kl}$ . The tensor  $\mathbf{L}$  varies with deformation, but is independent of  $\mathbf{D}$ .



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## Reminder:

### Polar rates $\Leftrightarrow$ unrotated tensors

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For any tensor  $\mathbf{A}$  define “unrotation” operation  $\bar{A}_{ij} \equiv A_{mn} R_{mi} R_{nj}$

In direct notation,  $\bar{\mathbf{A}} \equiv \mathbf{R}^T \cdot \mathbf{A} \cdot \mathbf{R}$ . For a vector  $\mathbf{v}$ , the bar is  $\bar{v}_i = v_m R_{mi}$ .

For a fourth-order tensor  $\mathbf{U}$ ,  $\bar{U}_{ijkl} = U_{mnpq} R_{mi} R_{nj} R_{pk} R_{ql}$ , etc.

Polar rate:  $\overset{\circ}{A}_{ij} \equiv \dot{A}_{ij} - \Omega_{im} A_{mj} + A_{im} \Omega_{mj}$  where  $\Omega_{ij} \equiv \dot{R}_{im} R_{jm}$

A little manipulation shows that  $\overset{\circ}{\bar{\mathbf{A}}} = \bar{\overset{\circ}{\mathbf{A}}}$ . This equation defines the polar rate for a tensor of *any* order, and it makes it quite clear that

the equations  $\overset{\circ}{\sigma}_{ij} = L_{ijkl} D_{kl}$  and  $\overset{\circ}{\bar{\sigma}}_{ij} = \bar{L}_{ijkl} \bar{D}_{kl}$  are equivalent.



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## Key points to be covered in this talk

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- Using polar rates in a spatial equation is equivalent to using ordinary rates in the unrotated reference configuration.
- The polar rate isn't "better" *per se* than the Jaumann rate. If the polar stress rate is linear in  $\mathbf{D}$ , *so is the Jaumann stress rate!*
- The tangent stiffness *must* change with deformation. A "good objective rate" minimizes the amount that the tangent stiffness tensor must change. In this sense, polar is better than Jaumann for isotropic materials.
- Kirchhoff stress  $\tau_{ij} = J\sigma_{ij}$  should be used if the tangent stiffness is to be assumed major-symmetric.
- Objective rates all give the same answers for problems involving large rotation with *infinitesimal distortion*. They all give wrong answers for large distortion unless the tangent stiffness varies. The second Piola-Kirchhoff stiffness changes *least* for large distortion of anisotropic media.



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# Distortion vs. deformation

**Large distortion:** material changes shape.  $\Leftrightarrow$  The stretch tensor  $\underline{\mathbf{V}}$  is significantly anisotropic.

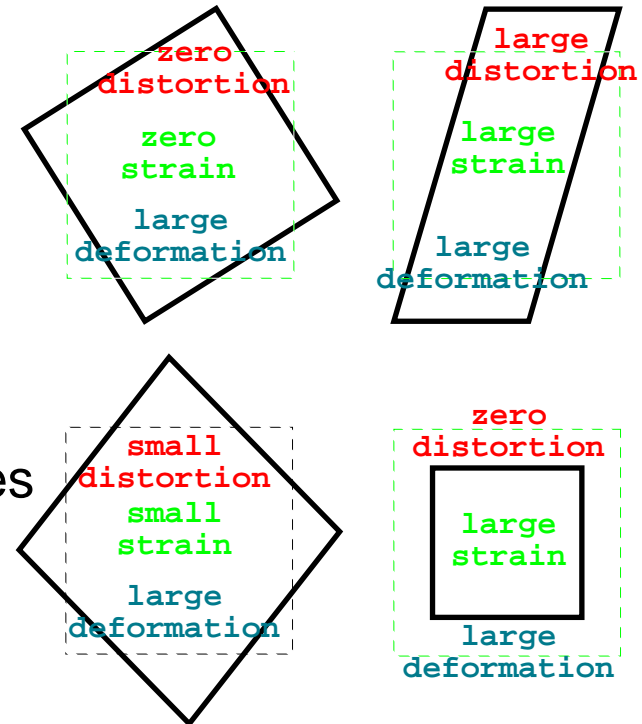
**Large strain:** material changes shape and/or size.  $\Leftrightarrow$  Small strain  $\underline{\mathbf{V}}$  differs significantly from the identity tensor  $\underline{\mathbf{I}}$ .

**Large deformation:** material changes shape, size, and/or orientation.

$\Leftrightarrow$  The deformation gradient tensor  $\underline{\mathbf{F}}$  is significantly different from the identity tensor  $\underline{\mathbf{I}}$ . Equivalently, displacement gradients are large.

**Large distortion  $\Rightarrow$  large strain  $\Rightarrow$  large deformation**

**Converse is false!** Models that ostensibly apply for large deformation might give unsatisfactory answers for large distortion.



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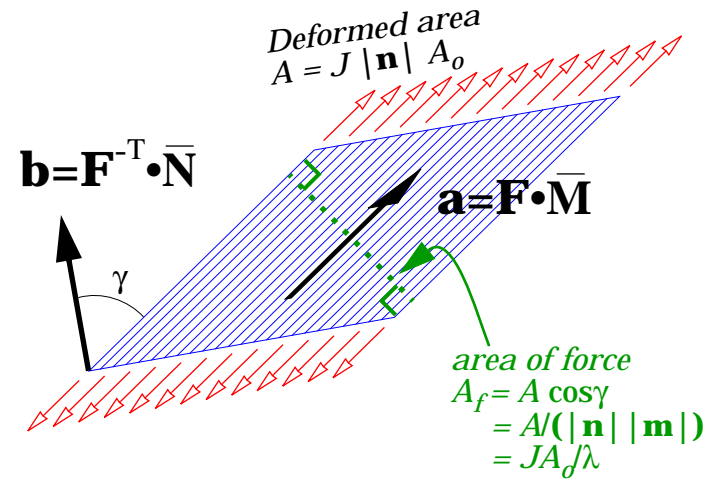
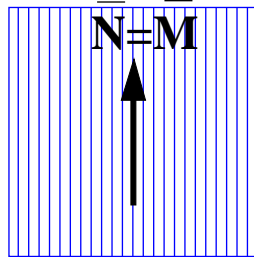




# Unrotated tensors will not adequately generalize a small distortion model.

Proof by counterexample:

Initial Cross-sectional area  $A_0$



Force in single fiber =  $F(\lambda)$

Exact solution for the Cauchy stress:  $\sigma_{ij} = \frac{G(\lambda)}{J} a_i a_j$

where  $G(\lambda) \equiv v_o \frac{F(\lambda)}{\lambda}$ ,  $\lambda \equiv \sqrt{\bar{\mathbf{M}} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \bar{\mathbf{M}}} = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ , and  $J \equiv \det \mathbf{F}$

(Note:  $\mathbf{N}$  and  $\mathbf{M}$  are unit vectors, but neither  $\mathbf{n}$  nor  $\mathbf{m}$  are unit vectors.)



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## Other expressions for the EXACT solution

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Unrotated stress: (note  $a_i = F_{ij}\bar{M}_j = R_{ik}\bar{V}_{kj}\bar{M}_j$ . Thus  $\bar{a}_i = \bar{V}_{ij}\bar{M}_j$ )

$$\bar{\sigma}_{ij} = \frac{G(\lambda)}{J} \bar{a}_i \bar{a}_j \quad \Rightarrow \quad \dot{\bar{\sigma}}_{ij} = \bar{L}_{ijkl} \bar{D}_{kl}$$

Second Piola-Kirchhoff stress:  $\bar{s}_{ij} = J F_{ik}^{-1} \sigma_{kl} F_{lj}^{-T} = J \bar{V}_{ik}^{-1} \bar{\sigma}_{kl} \bar{V}_{lj}^{-1}$

$$\bar{s}_{ij} = G(\lambda) \bar{M}_i \bar{M}_j \quad \Rightarrow \quad \dot{\bar{s}}_{ij} = \bar{E}_{ijkl} \dot{\bar{\epsilon}}_{kl}$$

The Cauchy stiffness  $\bar{L}_{ijkl}$  is a complicated function of the deformation. The PK2 stiffness  $\bar{E}_{ijkl}$  is intuitively simple in form:

$$\bar{E}_{ijkl} = \frac{G'(\lambda)}{\lambda} \bar{M}_i \bar{M}_j \bar{M}_k \bar{M}_l$$



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## Both stiffnesses are identical for small distortion

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When the material strains are small,

$$\bar{E}_{ijkl}^0 = \bar{L}_{ijkl}^0 = \nu_o F'(1) \bar{M}_i \bar{M}_j \bar{M}_k \bar{M}_l$$

The exact solution proves that the stiffnesses *must* change over time. In other words, the stress-strain relation must be nonlinear.

Question:

How bad are the answers if we nevertheless take the stiffnesses to be constant?



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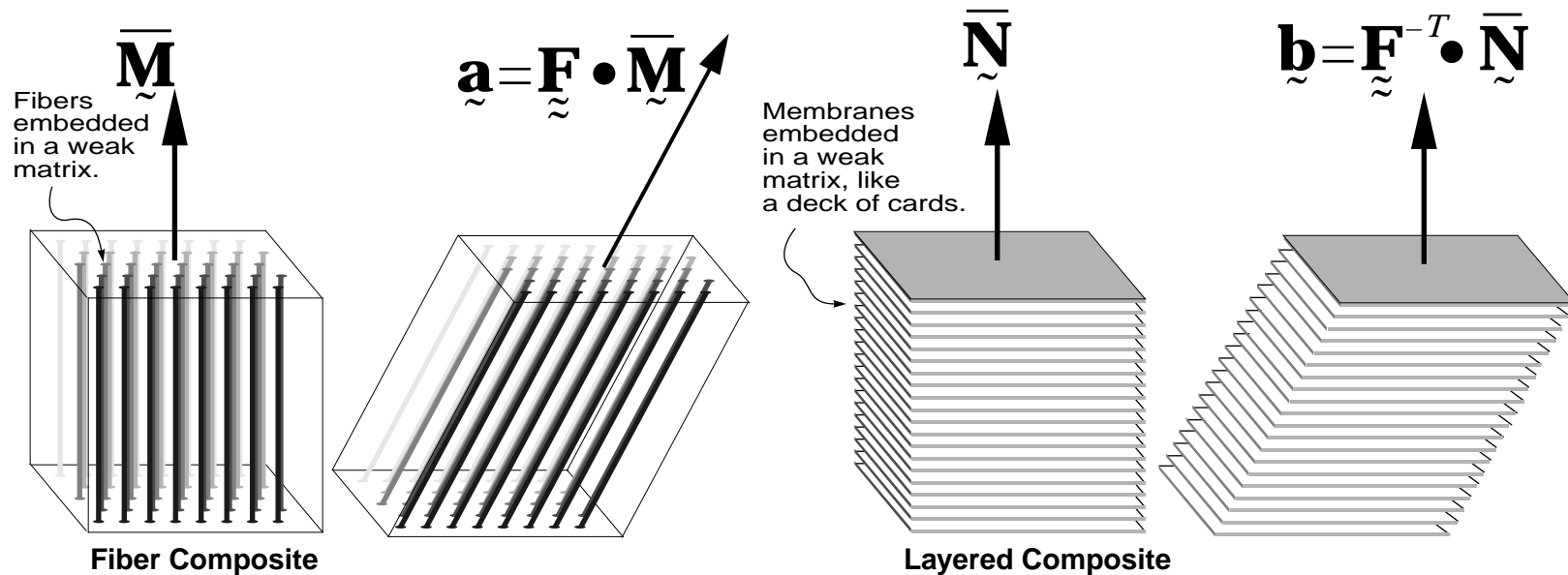




# Consequences of a constant Cauchy stiffness

Suppose we assume  $\bar{L}_{ijkl} = \bar{L}_{ijkl}^0 = \nu_o F'(1) \bar{M}_i \bar{M}_j \bar{M}_k \bar{M}_l = \text{constant}$ .

Then the magnitude will be wrong. More importantly, the unrotated stress will be uniaxial in the *wrong* direction (Zheng, 1992).



On the other hand, *at least the calculation will be robust*. Whoopee.



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## Consequences of a constant PK2 stiffness

Recall:  $\bar{s}_{ij} = G(\lambda)\bar{M}_i\bar{M}_j \Rightarrow \dot{s}_{ij} = \bar{E}_{ijkl}\dot{\epsilon}_{kl}$

where  $\bar{E}_{ijkl} = \frac{G'(\lambda)}{\lambda}\bar{M}_i\bar{M}_j\bar{M}_k\bar{M}_l$ .

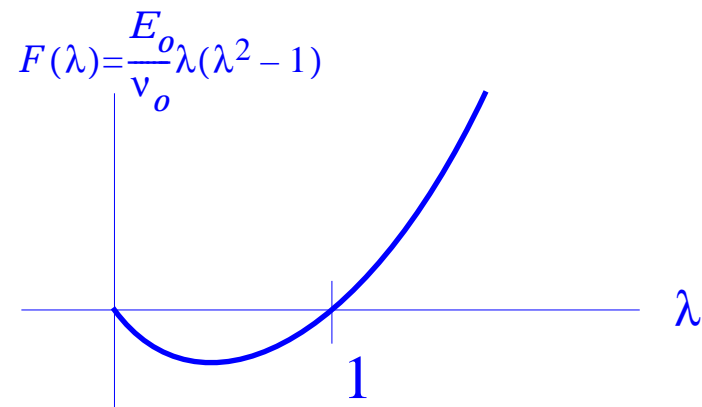
Suppose we assume  $\bar{E}_{ijkl} = \bar{E}_{ijkl}^0 = G'(1)\bar{M}_i\bar{M}_j\bar{M}_k\bar{M}_l = \text{constant}$ .

Then the stress will always be uniaxial in the correct direction, but the magnitude will be wrong.

**How bad is the error?**

Taking  $\bar{E}_{ijkl} = \bar{E}_{ijkl}^0$  is equivalent to taking  $G'(\lambda)/\lambda = \text{constant}$ .

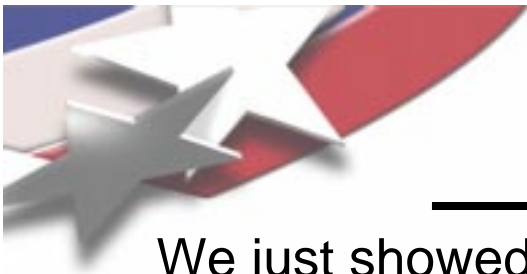
This implies a highly nonphysical *unstable* single-fiber force function...



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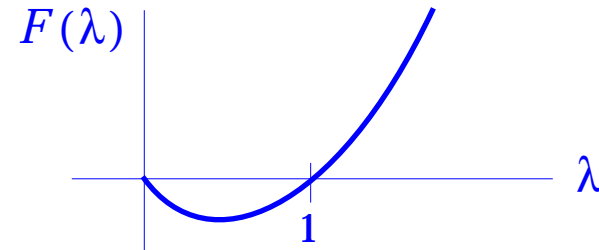
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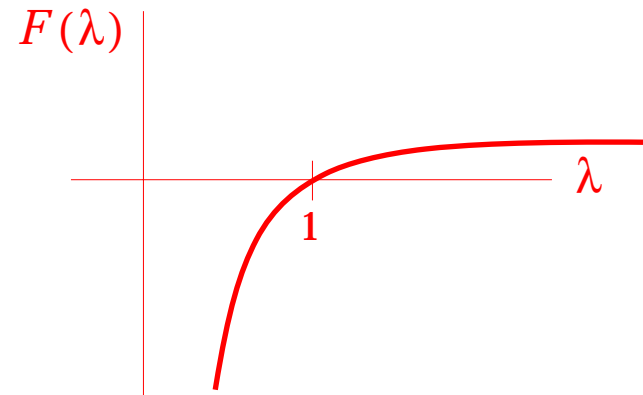
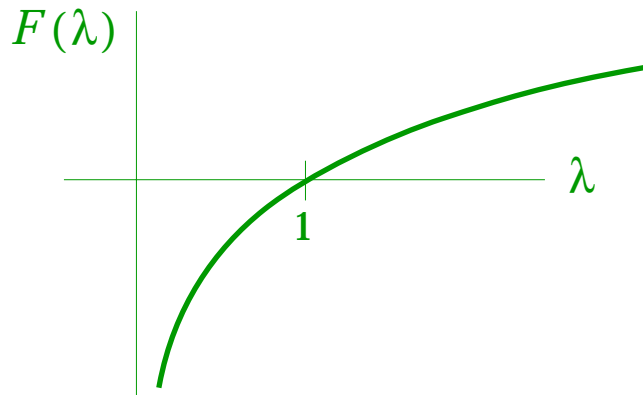


# Why does Cauchy “win” in practice?

We just showed that **constant PK2 stiffness** ( $\bar{E}_{ijkl}$  in  $\dot{s}_{ij} = \bar{E}_{ijkl}\dot{\epsilon}_{kl}$ ) implies...



In *uniaxial strain* **constant Cauchy stiffness** ( $\bar{L}_{ijkl}$  in  $\dot{\sigma}_{ij} = \bar{L}_{ijkl}\bar{D}_{kl}$ ) or **constant Kirchhoff stiffness** ( $\bar{\xi}_{ijkl}$  in  $\dot{\tau}_{ij} = \bar{\xi}_{ijkl}\bar{D}_{kl}$ ) implies...



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## Conclusions

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- The polar rate isn't "better" than the Jaumann rate. The culprit is an erroneous assumption of constant stiffness.
- The tangent stiffness *must* change with deformation.
- A "good choice" for stress and strain measures (or their rates) is deemed to be the choice that involves the least amount of change in the stiffness tensor needed to obtain physically sensible results.
- Considering the exact solution for fibers-in-air, the correct Cauchy stiffness was found to be extremely complicated function of deformation. Furthermore it is not even major symmetric.
- *Wrongly* using a constant Cauchy stiffness leads to errors in stress magnitude and direction — but, by golly, it's stable.
- The PK2 stiffness is in the correct direction, but wrongly taking it constant not only gives bad answers, but can lead to numerical instabilities.
- Bottom Line: constant stiffness is wrong. Even though the PK2 stiffness moduli *must* vary with deformation (to avoid instability in compression), the principal directions of the PK2 stiffness remain constant. It is therefore preferable over the Cauchy stiffness because the exact solution for the PK2 stiffness changes in the least complicated manner.
- Now research is needed to approximate the nonlinear PK2 moduli.



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# Spectral Expression for Nonlinear Stiffness

The exact PK2 fiber stiffness is of the form

$$\bar{E}_{ijkl} = H(\lambda) E^0 \bar{A}_{ij} \bar{A}_{kl},$$

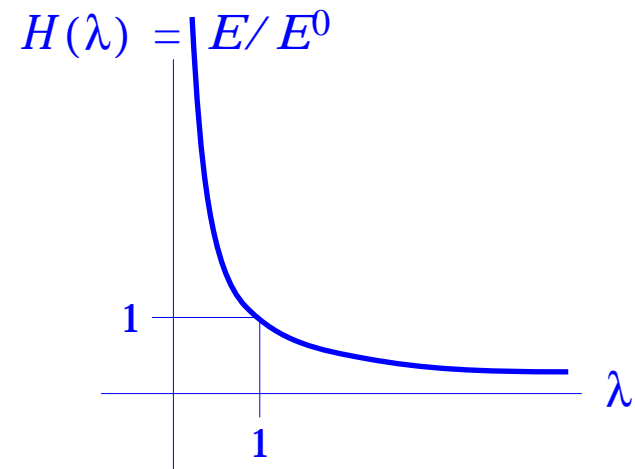
where  $\bar{A}_{ij} \equiv \bar{M}_i \bar{M}_j$  is an unvarying eigentensor,

The function

$$H(\lambda) = \frac{\lambda F'(\lambda) - F(\lambda)}{\lambda^3 F'(1)}$$

characterizes the strain nonlinearity.

IDEA: hypothesize that the PK2 stiffness for general materials will have unchanging eigentensors and the eigenvalues will vary with stretch similar to the fiber material.



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## Constant anisotropy, but nonlinear moduli

We propose a nonlinear PK2 modulus tensor  $\bar{C}_{ijkl}$  defined by

$$\bar{C}_{ijkl} = c_1(\varepsilon_1) \bar{B}_{ijkl}^1 + c_2(\varepsilon_2) \bar{B}_{ijkl}^2 + c_3(\varepsilon_2) \bar{B}_{ijkl}^3 + c_4(\varepsilon_4) \bar{B}_{ijkl}^4 + c_5(\varepsilon_5) \bar{B}_{ijkl}^5$$

where  $\varepsilon^\alpha \equiv \sqrt{\bar{\varepsilon}_{ij} \bar{B}_{ijpq}^\alpha \bar{B}_{pqkl}^\alpha \bar{\varepsilon}_{kl}}$  and the five  $c_i$  are material functions for which  $c_i(0)$  must equal the appropriate modulus measured under small distortions. The transverse basis tensors are

$$\bar{B}_{ijkl}^1 = \bar{M}_i \bar{M}_j \bar{M}_k \bar{M}_l \quad \bar{B}_{ijkl}^2 = \delta_{ij} \delta_{kl} - \bar{M}_i \bar{M}_j \delta_{kl} - \delta_{ij} \bar{M}_k \bar{M}_l + \bar{M}_i \bar{M}_j \bar{M}_k \bar{M}_l$$

$$\bar{B}_{ijkl}^3 = \bar{M}_i \bar{M}_j \delta_{kl} + \bar{M}_k \bar{M}_l \delta_{ij} - 2 \bar{M}_i \bar{M}_j \bar{M}_k \bar{M}_l$$

$$\bar{B}_{ijkl}^4 = \frac{1}{2} (\bar{M}_i \bar{M}_k \delta_{jl} + \bar{M}_j \bar{M}_l \delta_{ik} + \bar{M}_i \bar{M}_l \delta_{jk} + \bar{M}_j \bar{M}_k \delta_{il}) - 2 \bar{M}_i \bar{M}_j \bar{M}_k \bar{M}_l$$

$$\bar{B}_{ijkl}^5 = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \bar{M}_i \bar{M}_j \bar{M}_k \bar{M}_l - \frac{1}{2} (\delta_{ik} \bar{M}_j \bar{M}_l + \delta_{il} \bar{M}_j \bar{M}_k + \delta_{jk} \bar{M}_i \bar{M}_l + \delta_{jl} \bar{M}_i \bar{M}_k)$$

Defining the equivalent stretch as  $\lambda^\alpha \equiv \sqrt{2\varepsilon^\alpha + 1}$ , we can construct the function  $c_1$  to match the exact fibers-in-air solution.



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## General nonlinear PK2 stiffness

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Proposal:  $\bar{\mathbf{s}} = \sum_{K=1}^N E_K^0 f(\tilde{\bar{\boldsymbol{\varepsilon}}}_K)$  where  $E_K^0$  is the  $K^{\text{th}}$  eigenvalue of the initial (small strain) stiffness and  $\tilde{\bar{\boldsymbol{\varepsilon}}}_K = \bar{\mathbf{P}}^K : \bar{\boldsymbol{\varepsilon}}$  is the projection of the (large) strain onto the  $K^{\text{th}}$  eigenplane defined by the eigenprojector

$$\bar{\mathbf{P}}^K = \sum_{\alpha=1}^{m_K} \mathbf{A}^K_{\alpha} \otimes \mathbf{A}^K_{\alpha}.$$

Here,  $m_K$  is the multiplicity.

The function  $f(\tilde{\bar{\boldsymbol{\varepsilon}}}_K)$  would impose the nonlinear strain dependence, and would be required to approximate  $\tilde{\bar{\boldsymbol{\varepsilon}}}_K$  for small strains.



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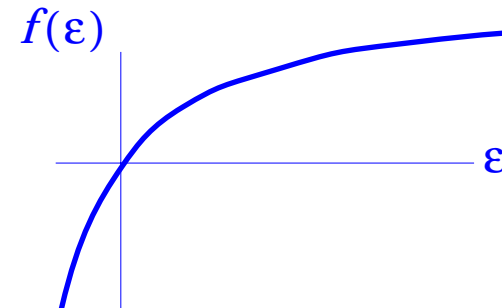
## A cheaper approximation function $f$

Consider

$$f(\bar{\varepsilon}) = \frac{1}{2}(\mathbf{I} - (2\bar{\varepsilon} + \mathbf{I})^{-1})$$

### Advantages:

- Direct notation — easier to evaluate.
- $f(\bar{\varepsilon}) \approx \bar{\varepsilon}$  for small  $\bar{\varepsilon}$ .
- Concave down.
- Infinite at extreme strain limits.



**Disadvantage:** this function is an approximation. It is incapable of exactly matching the analytical fiber solution.



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## An consistent nonlinear function $f$

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One proposal for the function  $f(\underline{\varepsilon})$  for any second-order tensor  $\underline{\varepsilon}$ :

1. Compute the principal strains  $\varepsilon_k$  and eigenprojectors  $\mathbf{p}_k$  of the tensor  $\underline{\varepsilon}$ .
2. Compute the “principal stretches”  $\lambda_k = \sqrt{2\varepsilon_k + 1}$  .

3. Then  $f(\underline{\tilde{\varepsilon}}) = \sum_{k=1}^m H(\lambda_k)\varepsilon_k\mathbf{p}_k$  where  $m$  is # distinct eigenvalues.

The function  $H$  would be an appropriate intensifier, preferably measured in the lab. In absence of data,  $H$  could be the function computed for the fiber material using, say, logarithmic fiber force:

$$H(\lambda) = \frac{\lambda F'(\lambda) - F(\lambda)}{\lambda^3 F'(1)} = \frac{1 - \ln \lambda}{\lambda^3}$$

Note that  $f(\underline{\tilde{\varepsilon}}) \approx \underline{\tilde{\varepsilon}}$  for small  $\varepsilon_k$ , as required.

In the absence of data,  $H$  could be the function computed for the fiber material using, say, logarithmic fiber force.



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## EXAMPLE

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Consider elastic isotropy. Then the initial stiffness is isotropic and there are two projected strains:

$$\bar{\tilde{\epsilon}}^{\text{iso}} \equiv \frac{1}{3}(\text{tr}\bar{\tilde{\epsilon}})\mathbf{I} \quad \text{and} \quad \bar{\tilde{\epsilon}}^{\text{dev}} \equiv \bar{\tilde{\epsilon}} - \frac{1}{3}(\text{tr}\bar{\tilde{\epsilon}})\mathbf{I}$$

So the PK2 stress is given by

$$\bar{\mathbf{s}} = 2Gf(\bar{\tilde{\epsilon}}^{\text{dev}}) + 3Kf(\bar{\tilde{\epsilon}}^{\text{iso}})$$



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## CASE: Uniaxial strain.

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \tilde{\bar{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \tilde{\bar{\boldsymbol{\varepsilon}}}^{\text{iso}} \equiv \frac{\varepsilon}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \tilde{\bar{\boldsymbol{\varepsilon}}}^{\text{dev}} = \frac{\varepsilon}{3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

where  $\varepsilon = \frac{1}{2}(\lambda^2 - 1)$

Then

$$\bar{\mathbf{s}} = 2G \begin{bmatrix} f(2\varepsilon/3) & 0 & 0 \\ 0 & f(-\varepsilon/3) & 0 \\ 0 & 0 & f(-\varepsilon/3) \end{bmatrix} + 3Kf\left(\frac{\varepsilon}{3}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{and } \bar{\boldsymbol{\sigma}} = 2G \begin{bmatrix} \lambda f(2\varepsilon/3) & 0 & 0 \\ 0 & f(-\varepsilon/3)/\lambda & 0 \\ 0 & 0 & f(-\varepsilon/3)/\lambda \end{bmatrix} + 3Kf\left(\frac{\varepsilon}{3}\right) \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix}.$$



Reference: Brannon, R.M., Acta Mech, Vol. 129, 1998.



## CASE: Pure shear.

$$\mathbf{F} = \begin{bmatrix} 1 & 2\varepsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{\varepsilon} = \begin{bmatrix} 0 & \varepsilon & 0 \\ \varepsilon & 2\varepsilon^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \tilde{\varepsilon}^{\text{iso}} \equiv \frac{2\varepsilon^2}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \tilde{\varepsilon}^{\text{dev}} = \begin{bmatrix} -2\varepsilon^2/3 & \varepsilon & 0 \\ \varepsilon & 4\varepsilon^2/3 & 0 \\ 0 & 0 & -2\varepsilon^2/3 \end{bmatrix}$$

$$\bar{\mathbf{s}} = 2G \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix} + 3Kf\left(\frac{2\varepsilon^2}{3}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

-- This viewgraph is unfinished --  
(not shown at presentation)



Reference: Brannon, R.M., Acta Mech, Vol. 129, 1998.

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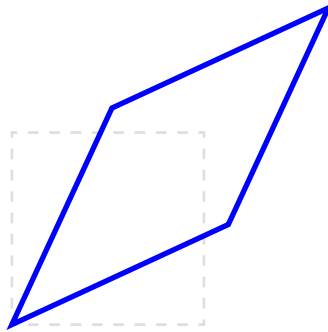
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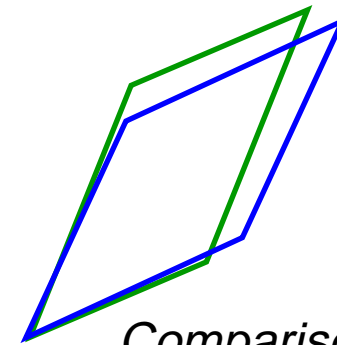
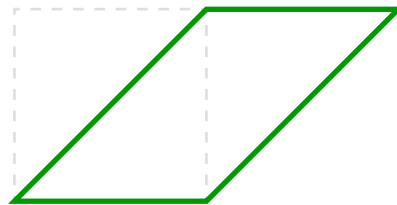
## What is a strain? What is a strain rate?

These two deformation gradients represent quite different motions:

$$\mathbf{F} = \begin{bmatrix} \cosh \epsilon & \sinh \epsilon & 0 \\ \sinh \epsilon & \cosh \epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathbf{F} = \begin{bmatrix} 1 & 2\epsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



*Comparison  
of reference  
stretches*

They have different stretches and different velocity gradients ( $\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}$ ) but they both have the *same* “rate of deformation” tensor  $\mathbf{D} \equiv \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$ .



Reference: Brannon, R.M., Acta Mech, Vol. 129, 1998.

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