# CLOSED FORM (purely algebraic) spectral decomposition of symmetric second-order tensors (class $V_3^2$)

This section begins with definitions and derivations that will ultimately lead to a method for finding eigenvalues and eigenvectors analytically.

RECOMMENDATION: Go ahead and shamelessly peek at the final result on page 400 and the **example on page 749.** Once familiar with the process, come back here to see why it works.

In mechanics, one frequently desires the eigenvalues and eigenvectors for second-order tensors. If $[A]$ is the $3\times 3$ lab component matrix associated with a symmetric tensor $\boldsymbol{\underset{\sim}{A}}$, then the eigenvalues are solutions to

$$\lambda^3 - I_1^A\lambda^2 + I_2^A\lambda - I_3^A = 0, \tag{21.98}$$

where $(I_1^A, I_2^A, I_3^A)$ are the characteristic invariants of $\boldsymbol{\underset{\sim}{A}}$. This is a cubic equation, so it has three solutions $(\lambda_1, \lambda_2, \lambda_3)$, some of which might be repeated. For the solution, most students simply use a solver function on their hand-held calculator (or some other sort of numerical approach). Like quadratic equations, a cubic equation can be solved directly in purely algebraic form. Unlike quadratic equations, however, the analytical solution to a cubic equation is comparatively ugly. If you look up the solution in a typical mathematics handbook [_], you will find some fairly inscrutable formulas that lack physical or geometric insight and might also involve complex numbers if applied verbatim (making it difficult to implement them in a code). In this section, we present a solution to this $3\times 3$ eigenvalue problem that is still pretty complicated, but has geometrical meaning and is guaranteed to avoid complex numbers if the starting matrix $[A]$ is real and symmetric.

***Eigenvalues.*** The first step toward a solution is to first compute the deviatoric part of $\boldsymbol{\underset{\sim}{A}}$:

$$\boldsymbol{\underset{\sim}{S}} \equiv \text{dev}\boldsymbol{\underset{\sim}{A}} = \boldsymbol{\underset{\sim}{A}} - \frac{I_1^A}{3}\boldsymbol{\underset{\sim}{I}}. \tag{21.99}$$

Physically, if $\boldsymbol{\underset{\sim}{A}}$ is a stress tensor, then $\boldsymbol{\underset{\sim}{S}}$ is a tensor measure of *shear* stress and $I_1^A/3$ is the mean stress (negative of pressure).

Importantly, $\boldsymbol{\underset{\sim}{A}}$ and $\boldsymbol{\underset{\sim}{S}}$ have the same eigenvectors. The eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ of $\boldsymbol{\underset{\sim}{A}}$ are related to the eigenvalues $(s_1, s_2, s_3)$ of $\boldsymbol{\underset{\sim}{S}}$ by

$$\lambda_k = s_k + I_1^A/3. \tag{21.100}$$

The characteristic equation for $\boldsymbol{\underset{\sim}{S}}$ is

$$s^3 - I_1^S s^2 + I_2^S s - I_3^S = 0, \tag{21.101}$$

where, as usual,

$$I_1^S = \text{tr}\boldsymbol{\underset{\sim}{S}} = 0, \qquad I_2^S = \text{tr}\boldsymbol{\underset{\sim}{S}}^C, \qquad I_3^S = \det\boldsymbol{\underset{\sim}{S}}. \tag{21.102}$$

The characteristic equation in Eq. (21.101) is still cubic, but it is simpler to solve than the Eq. (21.98). Because $\underset{\sim}{\boldsymbol{S}}$ is deviatoric, its trace $(I_1^S)$ is zero, so Eq. (21.101) has no $s^2$ term.

In what follows, it's useful to refer to the low, middle, and high eigenvalues. Therefore, in addition to letting $(s_1, s_2, s_3)$ denote the eigenvalues of $\underset{\sim}{\boldsymbol{S}}$, we will occasionally denote them by $(s_L, s_M, s_H)$, where $s_L \le s_M \le s_H$. Because $\underset{\sim}{\boldsymbol{S}}$ is deviatoric, we know that $s_L + s_M + s_H = 0$. Consequently, the smallest eigenvalue $s_L$ must be negative (or zero) and the largest eigenvalue $s_H$ must be positive (or zero). The middle eigenvalue $s_M$ can be of any sign. You can use this knowledge about the numerical signs of the eigenvalues to prove that $I_2^S$ will always be negative (or zero). When working with a *deviatoric* tensor, it is customary to define

$$J_2 = -I_2^S \quad \text{and} \quad J_3 = I_3^S. \tag{21.103}$$

These are called **mechanics invariants** because they are often employed in applied mechanics. Since $I_2^S$ is never positive, it follows that $J_2$ is never negative — this fact will be used later when $\sqrt{J_2}$ shows up (the result is real, not complex). The invariant $J_3$ can be of any sign. Because $\underset{\sim}{\boldsymbol{S}}$ is deviatoric (and therefore $s_1 + s_3 + s_3 = 0$), it can be shown that

$$J_2 = -\mathrm{tr}\underset{\sim}{\boldsymbol{S}}^C = \tfrac{1}{2}\mathrm{tr}\underset{\sim}{\boldsymbol{S}}^2 = \tfrac{1}{2}\left\|\underset{\sim}{\boldsymbol{S}}\right\|^2, \tag{21.103b}$$

$$J_3 = \det\underset{\sim}{\boldsymbol{S}} = \tfrac{1}{3}\mathrm{tr}\underset{\sim}{\boldsymbol{S}}^3. \tag{21.103c}$$

In our efforts to solve the original eigenproblem of Eq. (21.98), we have switched from the standard set of characteristic invariants $(I_1^A, I_2^A, I_3^A)$ to a different set of invariants $(I_1^A, J_2, J_3)$. The transformation is invertible (i.e., given one triplet of invariants, you can compute the other triplet). Therefore no information has been lost. To compute the new set of invariants directly from components, the following formulas may be used

$$I_1^A = A_{11} + A_{22} + A_{33}, \tag{21.104a}$$

$$J_2 = \tfrac{1}{2}(S_{11}^2 + S_{22}^2 + S_{33}^2) + (S_{12}^2 + S_{23}^2 + S_{31}^2), \tag{21.104b}$$

$$J_3 = S_{11}S_{22}S_{33} + 2S_{12}S_{23}S_{31} - (S_{11}S_{23}^2 + S_{22}S_{31}^2 + S_{33}S_{12}^2). \tag{21.104c}$$

The last two of these formulas have been simplified to take advantage of the fact that $\underset{\sim}{\boldsymbol{S}}$ is symmetric and deviatoric.

In terms of the new symbols for the invariants, Eq. (21.101) may be written

$$s^3 - J_2 s - J_3 = 0. \tag{21.105}$$

This is still a cubic equation, but at least it is simpler. To solve this equation, we are going to introduce *yet another* triplet of invariants — called **Lode** coordinates. As explained later, eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ of $\underset{\sim}{\boldsymbol{A}}$ can be regarded as Cartesian components of a "pseudo-vector" in a 3D space. The eigenvalues $(s_1, s_2, s_3)$ of $\underset{\sim}{\boldsymbol{S}}$ are also like coordinates of a point in this space, but since $s_1 + s_2 + s_3 = 0$, this point must lie in the plane that is perpendicular to the [111] direction and passes through the origin. This plane is called the **pi-plane** and any plane parallel to it (but not containing the origin) is called an **octahe-**

**dral plane**. The Lode coordinates are mathematically isomorphic to (i.e., geometrically similar to) cylindrical coordinates defined such that the $z$-symmetry axis is aligned with the [111] direction and the other two axes lie within the pi-plane. The Lode coordinate triplet is found from the invariant triplet in Eq. (21.104) by

$$r = \|\underset{\sim}{S}\| = \sqrt{\underset{\sim}{S}:\underset{\sim}{S}} \qquad \text{= magnitude of S = mag of part of A perpendicular to [111] direction} \tag{21.106a}$$

$$\sin 3\theta = \hat{\underset{\sim}{S}}:\hat{\underset{\sim}{T}} \qquad \text{= cosine of angle b/w S and T} \qquad \textit{Use principal ArcSine for } \theta^{*} \tag{21.106b}$$

$$z = \underset{\sim}{A}:\hat{\underset{\sim}{I}} \qquad \text{= component of A in the [111] direction} \tag{21.106c}$$

Here,

$$\hat{\underset{\sim}{I}} = \frac{\underset{\sim}{I}}{\|\underset{\sim}{I}\|} = \frac{\underset{\sim}{I}}{\sqrt{3}} \tag{21.107a}$$

$$\underset{\sim}{S} \equiv \text{dev}\underset{\sim}{A} = \underset{\sim}{A} - \tfrac{1}{3}\text{tr}(\underset{\sim}{A})\underset{\sim}{I} \qquad \hat{\underset{\sim}{S}} = \underset{\sim}{S}/\|\underset{\sim}{S}\| \tag{21.107b}$$

$$\underset{\sim}{T} = \text{dev}\underset{\sim}{S}^2 \qquad \hat{\underset{\sim}{T}} = \underset{\sim}{T}/\|\underset{\sim}{T}\| \tag{21.107c}$$

The Lode invariants may be computed directly from mechanics invariants $(I_1, J_2, J_3)$ by

$$r = +\sqrt{2J_2} \tag{21.108a}$$

$$\sin 3\theta = \frac{J_3}{2}\left(\frac{3}{J_2}\right)^{-3/2} = 3\sqrt{6}\det\hat{\underset{\sim}{S}}, \qquad \text{where } \hat{\underset{\sim}{S}} \equiv \frac{\underset{\sim}{S}}{r} \tag{21.108b}$$

$$z = \frac{I_1}{\sqrt{3}}. \tag{21.108c}$$

Strictly speaking, this new invariant triplet is not $(r, \theta, z)$. It is actually $(r, \sin 3\theta, z)$. Recalling that $J_2 \geq 0$, note that $r$ will be real, and it is taken without loss in generality to be positive. Because $\underset{\sim}{S}$ is deviatoric, it can be verified that the right-hand-side of (21.108b) will fall in the range from -1 to 1, so it is indeed permissible to denote this combination by the sine of an angle. If the second equation is solved to assign a value to $\theta$, the principal ArcSin should be used so that $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$. †

* The principal ArcSin is always in the range $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$. Any angle $\Theta$ outside this range may be transformed to this range by $\theta = \frac{1}{3}\text{ArcSin}[\sin(3\Theta)]$. This transformation is especially useful in plotting octahedral profiles where the polar angle in such plots is $\Theta$, which varies from 0 to $2\pi$, but the formula for the radius depends on $\theta$. When $\Theta$ is transformed to $\theta$, which is then substituted into $r(\theta)$, the resulting octahedral profile will have the desired $120°$ rotational and reflective symmetry.

† The information covered in this section is applicable to any symmetric second-order tensor $\underset{\sim}{A}$ referenced to 3D space. If, however, this is a *stress* tensor, then $z$ is proportional to mean stress (-pressure), and $r$ is a measure of equivalent shear stress. When $\theta = \pi/6$, the stress is said to be "triaxial extension" (MMH). When $\theta = -\pi/6$, the stress is "triaxial compression" (LMM). The term "triaxial" is a bit of a misnomer -- these are the stress states that are axisymmetric so that two principal stresses (lateral) are equal and different from the remaining (axial) stress. For MMH, the axial stress is larger (more tensile) than the lateral stress. For LMM, the axial stress is smaller (more compressive) than the lateral stress.

The three solutions $(s_1, s_2, s_3)$ to the characteristic equation could be regarded as a point in a 3D space. Geometrically, the constraint that $s_1 + s_2 + s_3 = 0$ defines a plane (called the "pi-plane") whose unit normal points in the [111] direction. Consequently, the solutions we seek must lie in this plane. The *ordered* eigenvalues must fall in a $60°$ sextant in the plane for which $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$. In terms of a cylindrical coordinate system centered about the [111] direction, any point in this sextant may be represented by Cartesian coordinates $(r\cos\theta, r\sin\theta)$. If the solutions $(s_1, s_2, s_3)$ to Eq. 21.105 were known, note that

$$J_2 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2) \qquad \text{and} \qquad J_3 = s_1 s_2 s_3 . \tag{21.109}$$

Note that $s_1$ doesn't play a role that is structurally different from $s_2$ and $s_3$. Swapping any two eigenvalues will not change the final result. This symmetry implies that if $(s_1, s_2, s_3)$ produces certain values for $J_2$ and $J_3$, then so will $(s_2, s_1, s_3)$ and so will $(s_3, s_2, s_1)$, etc. Geometrically, this means that there is a family of points in the pi-plane corresponding to the same values of $J_2$ and $J_3$ and therefore corresponding to the same solutions of the characteristic equation. Since all of these points have the same value of $J_2$, then they must all have the same value of $s_1^2 + s_2^2 + s_3^2$. In other words, they must all be equidistant from the origin. Moreover, because of the swapping property, they must be separated by angles of $120°$ in the pi-plane. Thus, if one solution is found, then the others two can be determined by simple rotations in the pi-plane. All of this qualitative hand-waving suggests that we should look for a solution to Eq. (21.105) in the form $s = \rho\cos\alpha$. Making such a substitution reveals that

$$\rho = \sqrt{\tfrac{2}{3}}r \qquad \text{and} \qquad \cos 3\alpha = \sin 3\theta \tag{21.110}$$

where $r$ and $\sin 3\theta$ are defined in Eq. (21.108). The ArcCosine of $\cos 3\alpha$ has three solutions, corresponding to our three solutions to the cubic equation. Ultimately, the distinction between these solution triplets is simply the *ordering* of the eigenvalues. The principal solution triplet is the one for which $s_L \le s_M \le s_H$. When eigenvalues are ordered this way, the point $(s_L, s_M, s_H)$ can be shown to have a Lode angle satisfying $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$. Thus, by using the principal ArcSin to determine $\theta$ from Eq. (21.108b), we are not only finding the three solutions to the characteristic solution, *we also are controlling their ordering*. Following through with this tedious analysis to determine $(s_L, s_M, s_H)$ and then applying (21.100) to determine the eigenvalues $(\lambda_L, \lambda_M, \lambda_H)$ for the original tensor $\underset{\sim}{A}$ ultimately gives the following *purely algebraic* formula expressed in terms of quantities that are always real if $\underset{\sim}{A}$ itself is real:

$$\lambda_H = \frac{z}{\sqrt{3}} + \frac{r}{\sqrt{2}}\left[\frac{\sin\theta}{\sqrt{3}} + \cos\theta\right] = \frac{z}{\sqrt{3}} + \sqrt{\frac{2}{3}}r\cos\left(\theta - \frac{\pi}{6}\right) \tag{21.111a}$$

$$\lambda_M = \frac{z}{\sqrt{3}} - \sqrt{\frac{2}{3}}r\sin\theta \tag{21.111b}$$

$$\lambda_L = \frac{z}{\sqrt{3}} - \frac{r}{\sqrt{2}}\left[\frac{\sin\theta}{\sqrt{3}} - \cos\theta\right] = \frac{z}{\sqrt{3}} - \sqrt{\frac{2}{3}}r\cos\left(\theta + \frac{\pi}{6}\right) \tag{21.111c}$$

The subscripts "H,M,L" stand for "high, middle, and low" because it can be shown that the above three formulas will always satisfy the inequality $\sigma_L \le \sigma_M \le \sigma_H$. Note from Eq. (21.111) that we have a triple root (i.e., the middle eigenvalue has multiplicity 3) if $r = 0$. We have a double root (i.e. the middle eigenvalue has multiplicity 2) if $\theta = \pm\frac{\pi}{6}$. Otherwise, the eigenvalues are all distinct.

Note that

$$\frac{\sigma_H - \sigma_M}{2} = \frac{r}{\sqrt{2}}\sin\left(\frac{\pi}{6} + \theta\right) \qquad \ge 0 \tag{21.112}$$

$$\frac{\sigma_M - \sigma_L}{2} = \frac{r}{\sqrt{2}}\sin\left(\frac{\pi}{6} - \theta\right) \qquad \ge 0. \tag{21.113}$$

The inequalities follow because $\theta$ always falls in the range $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$.

For materials modeling, it is often useful to define

$$R = \frac{\sigma_H - \sigma_L}{2} = \frac{r}{\sqrt{2}}\cos\theta \tag{21.114a}$$

$$M = \frac{\sigma_H + \sigma_L}{2} = \frac{z}{\sqrt{3}} + \frac{r}{\sqrt{6}}\sin\theta \tag{21.114b}$$

$$\eta = \frac{M - \sigma_M}{R} = \sqrt{3}\tan\theta. \tag{21.114c}$$

Here, $R$ is the radius of the largest Mohr's circle (see Fig. 27.11), $M$ is the center of the largest Mohr's circle, and $\eta$ is the eccentricity of the middle eigenvalue ($\eta$ varies from $-1$ in triaxial compression, to 0 in pure shear, to $+1$ in triaxial extension).

Being able to determine the eigenvalues (and their ordering) directly from invariants without a conventional eigenvalue analysis is extraordinarily useful because it permits also directly computing the associated eigen-*projectors*.

### Eigenspace projectors (from which eigenvectors may be found if desired).

As discussed on page 283, eigenvectors are not unique, but eigenspace projectors *are* unique. This section outlines methods for determining eigenspace projectors directly from knowledge of the eigenvalues and their multiplicities. Once eigenspace projectors are found, eigenvectors may (if desired) be found by Gram-Schmidt orthogonalization (GSO) of the columns of the eigenspace projectors (see page 219). This method for producing eigenvectors, by the way, results in eigenvectors with an "affinity" to the laboratory basis. If, for example, an eigenvalue is a double root, then two eigenvectors resulting from GSO of the associated projector will be aligned with projections of the lab base vectors $\underset{\sim}{\boldsymbol{E}}_1$ and $\underset{\sim}{\boldsymbol{E}}_2$ onto the 2D eigenplane (or, if either of these lab base vectors happens to be perpendic-

ular to the eigenplane, then one eigenvector will be the projection of $\underset{\sim}{\boldsymbol{E}}_3$ onto the plane). Even though eigenvectors may be determined from eigenspace projectors, doing so is not usually necessary in applications. Working directly with unique eigenspace projectors rather than non-unique eigenvectors is typically more efficient and certainly more elegant.

The inflated spectral decomposition (see page 286) is

$$\underset{\sim}{\boldsymbol{A}} = \lambda_L \underset{\sim}{\boldsymbol{P}}_L + \lambda_M \underset{\sim}{\boldsymbol{P}}_M + \lambda_H \underset{\sim}{\boldsymbol{P}}_H . \tag{21.115}$$

This shows that the eigenspace projectors, form a "basis" for the tensor $\underset{\sim}{\boldsymbol{A}}$. In fact, this expansion is structured a lot like the ordinary basis expansion for a vector $\underset{\sim}{\boldsymbol{v}} = v_1\underset{\sim}{\boldsymbol{e}}_1 + v_2\underset{\sim}{\boldsymbol{e}}_2 + v_3\underset{\sim}{\boldsymbol{e}}_3$. For this reason, the three eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ of the tensor $\underset{\sim}{\boldsymbol{A}}$ are often regarded as a point in a 3D space. This visualization of a *tensor* as if it were a *vector* has some isomorphisms (geometrical analogs). For example, the magnitude of the tensor, $\|\underset{\sim}{\boldsymbol{A}}\| = \lambda_L^2 + \lambda_M^2 + \lambda_H^2$, equals the magnitude of the pseudo vector, $\underset{\sim}{\boldsymbol{a}} = \lambda_1\underset{\sim}{\boldsymbol{e}}_1 + \lambda_2\underset{\sim}{\boldsymbol{e}}_2 + \lambda_3\underset{\sim}{\boldsymbol{e}}_3$. There are, however, some operations (such as the trace and determinant) that are defined only for tensors and there is no isomorphism to 3D space. Likewise, some ordinary vector operations (such as the cross product) can lack an isomorphism (i.e., a counterpart operator for tensors) if there are double or triple multiplicity eigenvalues, in which case $\underset{\sim}{\boldsymbol{P}}_L$ and/or $\underset{\sim}{\boldsymbol{P}}_H$ is a degenerate "spare" (see page 286).



Previously, we defined three invariants,

$$I_1 = \mathrm{tr}\underset{\sim}{\boldsymbol{A}}, \qquad J_2 = \frac{1}{2}\mathrm{tr}\underset{\sim}{\boldsymbol{S}}^2, \qquad J_3 = \frac{1}{3}\mathrm{tr}\underset{\sim}{\boldsymbol{S}}^3, \tag{21.116}$$

where (recall) $\underset{\sim}{\boldsymbol{S}}$ is the deviatoric part of $\underset{\sim}{\boldsymbol{A}}$. These invariants were used to define an alternative triplet of invariants, the Lode cylindrical coordinates $(r, \theta, z)$ defined in Eq. (21.108).

The Cayley-Hamilton theorem implies that

$$\underset{\sim}{\boldsymbol{S}}^3 = J_2\underset{\sim}{\boldsymbol{S}} + J_3\underset{\sim}{\boldsymbol{I}}, \tag{21.117}$$

which is useful for expressing the trace of higher powers of $\underset{\sim}{\boldsymbol{S}}$ in terms of the three fundamental invariants in Eq. (21.116). For example,

$$\mathrm{tr}\underset{\sim}{\boldsymbol{S}}^4 = \mathrm{tr}(J_2\underset{\sim}{\boldsymbol{S}}^2 + J_3\underset{\sim}{\boldsymbol{S}}) = 2J_2^2 . \tag{21.118}$$

The derivatives of the three invariants in Eq. (21.116) are

$$\frac{dI_1}{d\underset{\sim}{\boldsymbol{A}}} = \underset{\sim}{\boldsymbol{I}} \qquad \frac{dJ_2}{d\underset{\sim}{\boldsymbol{A}}} = \underset{\sim}{\boldsymbol{S}} \qquad \frac{dJ_3}{d\underset{\sim}{\boldsymbol{A}}} = \underset{\sim}{\boldsymbol{T}}, \tag{21.119}$$

where $\underset{\sim}{\boldsymbol{T}}$ is the deviatoric part of $\underset{\sim}{\boldsymbol{S}}^2$ and therefore*

$$\underset{\sim}{\boldsymbol{T}} = \underset{\sim}{\boldsymbol{S}}^2 - \frac{2}{3}J_2\underset{\sim}{\boldsymbol{I}}. \tag{21.120}$$

* This section apples to *any* symmetric second-order tensor $\underset{\sim}{\boldsymbol{A}}$ (of class $\mathcal{V}_3^2$). If, however, $\underset{\sim}{\boldsymbol{A}}$ happens to be a *stress* tensor $\underset{\sim}{\boldsymbol{\sigma}}$, then $\underset{\sim}{\boldsymbol{T}}$ is called the "Hill tensor" and $\underset{\sim}{\boldsymbol{S}}$ is the stress deviator.

The magnitudes of $\underset{\approx}{\boldsymbol{S}}$ and $\underset{\approx}{\boldsymbol{T}}$ are, respectively,

$$\left\|\underset{\approx}{\boldsymbol{S}}\right\| = r \qquad \text{and} \qquad \left\|\underset{\approx}{\boldsymbol{T}}\right\| = \frac{r^2}{\sqrt{6}}, \text{ where } r = \sqrt{2J_2}. \tag{21.121}$$

Unit tensors, $\hat{\underset{\approx}{\boldsymbol{S}}}$ and $\hat{\underset{\approx}{\boldsymbol{T}}}$, in the respective directions of $\underset{\approx}{\boldsymbol{S}}$ and $\underset{\approx}{\boldsymbol{T}}$ (which can be quite useful in applications beyond simply eigenproblems) can be computed quickly by*

$$\hat{\underset{\approx}{\boldsymbol{S}}} = \frac{\underset{\approx}{\boldsymbol{S}}}{r} \qquad \hat{\underset{\approx}{\boldsymbol{T}}} = \sqrt{6}\left[\hat{\underset{\approx}{\boldsymbol{S}}}^2 - \frac{1}{3}\underset{\approx}{\boldsymbol{I}}\right]. \tag{21.122}$$

Though not obvious, the tensors $\hat{\underset{\approx}{\boldsymbol{S}}}$ and $\hat{\underset{\approx}{\boldsymbol{T}}}$ are linearly independent if and only if all three eigenvalues are distinct (case LMH). For the double root MMH case, $\hat{\underset{\approx}{\boldsymbol{S}}} = \hat{\underset{\approx}{\boldsymbol{T}}}$. For the double root LMM case, $\hat{\underset{\approx}{\boldsymbol{S}}} = -\hat{\underset{\approx}{\boldsymbol{T}}}$. Of course, for the triple root MMM case, neither $\hat{\underset{\approx}{\boldsymbol{S}}}$ nor $\hat{\underset{\approx}{\boldsymbol{T}}}$ are uniquely defined (they may, if desired, be set to *any* deviatoric unit tensors; this statement is similar to saying that cylindrical base vectors $\underset{\sim}{\boldsymbol{e}}_r$ and $\underset{\sim}{\boldsymbol{e}}_\theta$ may be set to anything if the position vector happens to fall on the symmetry axis in cylindrical coordinates).

Individual unit eigenvectors are arbitrary within a multiple of $\pm 1$ and also possibly arbitrary in direction when there are repeated roots. Primary eigenspace *projectors*, on the other hand, are always unique. When eigenvalues are known, the eigenvectors never need to be found. The primary eigenspace projectors can be computed algebraically directly from the source tensor $\underset{\approx}{\boldsymbol{A}}$ and knowledge of eigenvalue multiplicity.

For the LMH case where the middle eigenvalue has multiplicity 1 (i.e., when a tensor has three *distinct* eigenvalues), the eigenspace projectors may be computed directly from the tensor itself by using Eq. (7.73):

$$\underset{\sim}{\boldsymbol{P}}_L = \frac{(\underset{\approx}{\boldsymbol{A}} - \lambda_M \underset{\approx}{\boldsymbol{I}}) \bullet (\underset{\approx}{\boldsymbol{A}} - \lambda_H \underset{\approx}{\boldsymbol{I}})}{(\lambda_L - \lambda_M)(\lambda_L - \lambda_H)}, \quad \underset{\sim}{\boldsymbol{P}}_M = \frac{(\underset{\approx}{\boldsymbol{A}} - \lambda_H \underset{\approx}{\boldsymbol{I}}) \bullet (\underset{\approx}{\boldsymbol{A}} - \lambda_L \underset{\approx}{\boldsymbol{I}})}{(\lambda_M - \lambda_H)(\lambda_M - \lambda_L)}, \quad \text{and} \quad \underset{\sim}{\boldsymbol{P}}_H = \frac{(\underset{\approx}{\boldsymbol{A}} - \lambda_L \underset{\approx}{\boldsymbol{I}}) \bullet (\underset{\approx}{\boldsymbol{A}} - \lambda_M \underset{\approx}{\boldsymbol{I}})}{(\lambda_H - \lambda_L)(\lambda_H - \lambda_M)} \tag{21.123}$$

For $3 \times 3$ matrices, these formulas can be made even simpler by using a cofactor operation. Specifically, for tensors of class $V_3^2$ having three distinct eigenvalues, the associated distinct and unique type-1 eigenspace projectors are

**LMH primary eigenspace projectors:**

$$\underset{\sim}{\boldsymbol{P}}_L = \frac{(\underset{\approx}{\boldsymbol{A}} - \lambda_L \underset{\approx}{\boldsymbol{I}})^C}{(\lambda_M - \lambda_L)(\lambda_H - \lambda_L)}, \quad \underset{\sim}{\boldsymbol{P}}_M = \frac{(\underset{\approx}{\boldsymbol{A}} - \lambda_M \underset{\approx}{\boldsymbol{I}})^C}{(\lambda_H - \lambda_M)(\lambda_L - \lambda_M)}, \quad \text{and} \quad \underset{\sim}{\boldsymbol{P}}_H = \frac{(\underset{\approx}{\boldsymbol{A}} - \lambda_H \underset{\approx}{\boldsymbol{I}})^C}{(\lambda_L - \lambda_H)(\lambda_M - \lambda_H)} \tag{21.124}$$

The cofactor operator "C" is defined on page 266.

Now consider a tensor that has an eigenvalue of *double* multiplicity (LMM or MMH). Again, Eq. (7.73) may be used to find the primary eigenspace projectors:

**LMM primary eigenspace projectors:**

$$\underset{\approx}{\boldsymbol{P}}_M = \frac{(\underset{\approx}{\boldsymbol{A}} - \lambda_L \underset{\approx}{\boldsymbol{I}})}{(\lambda_M - \lambda_L)} \quad \text{and} \quad \underset{\approx}{\boldsymbol{P}}_L = \frac{(\underset{\approx}{\boldsymbol{A}} - \lambda_M \underset{\approx}{\boldsymbol{I}})}{(\lambda_L - \lambda_M)} = \underset{\approx}{\boldsymbol{I}} - \underset{\approx}{\boldsymbol{P}}_M \tag{21.125}$$

or

* These formulas are merely dividing the tensors by their own magnitudes.

**LMM primary projectors:**

$$\underset{\approx}{\boldsymbol{P}}_M = \frac{(\underset{\approx}{\boldsymbol{A}} - \lambda_H \underset{\approx}{\boldsymbol{I}})}{(\lambda_M - \lambda_H)} \quad \text{and} \quad \underset{\approx}{\boldsymbol{P}}_H = \frac{(\underset{\approx}{\boldsymbol{A}} - \lambda_M \underset{\approx}{\boldsymbol{I}})}{(\lambda_H - \lambda_M)} = \underset{\approx}{\boldsymbol{I}} - \underset{\approx}{\boldsymbol{P}}_M \tag{21.126}$$

Consider finally a tensor $\underset{\approx}{\boldsymbol{A}}$ that has only one distinct eigenvalue. It must therefore have *triple* multiplicity. In this MMM case, the tensor must be isotropic (i.e., a multiple of the identity tensor), and *any* vector is an eigenvector. The primary projector is simply the identity tensor.

**MMM primary projector:**

$$\underset{\approx}{\boldsymbol{P}}_M = \underset{\approx}{\boldsymbol{I}}. \tag{21.127}$$

For cases involving multiple roots, Eq. (15.118) may be used to *define* spare degenerate projectors so that the inflated spectral decomposition may be written

$$\underset{\approx}{\boldsymbol{A}} = \sum_{k=1}^{3} \lambda_k \underset{\approx}{\boldsymbol{P}}_k. \tag{21.128}$$

The number of nontrivial (nonzero) terms in this summation of eigenvalues $\lambda_k$ times associated eigenspace projectors $\underset{\approx}{\boldsymbol{P}}_k$ will equal the number of *distinct* eigenvalues. Because we are discussing symmetric tensors, each eigenspace projector $\underset{\approx}{\boldsymbol{P}}_k$ is a closest-point projector. Therefore $\underset{\approx}{\boldsymbol{P}}_k^T = \underset{\approx}{\boldsymbol{P}}_k$, $\underset{\approx}{\boldsymbol{P}}_k \bullet \underset{\approx}{\boldsymbol{P}}_k = \underset{\approx}{\boldsymbol{P}}_k$ (no sum on $k$) and $\underset{\approx}{\boldsymbol{P}}_m \bullet \underset{\approx}{\boldsymbol{P}}_n = 0$ if $m \neq n$. Moreover $\underset{\approx}{\boldsymbol{P}}_m : \underset{\approx}{\boldsymbol{P}}_n = 0$ if $m \neq n$ and $\underset{\approx}{\boldsymbol{P}}_k : \underset{\approx}{\boldsymbol{P}}_k$ equals the multiplicity of the $k^{\text{th}}$ eigenvalue. Finally, the sum of all eigenprojectors always equals the identity tensor.*

In this discussion of eigen-projectors and spectral expansions, the number of distinct eigenvalues plays a crucial role. Is there a way to determine the number of distinct eigenvalues *without* needing to use Eq. (21.111) to actually *compute* (and then compare) the eigenvalues? Yes, as follows:

***If*** $r = 0$***, then there is only one distinct eigenvalue (MMM)***

***Else, if*** $\sin(3\theta) = 1$***, then there are two distinct eigenvalues,*** $\sigma_H$ ***and*** $\sigma_M$ ***(MMH)***

***Else, if*** $\sin(3\theta) = -1$***, then there are two distinct eigenvalues,*** $\sigma_L$ ***and*** $\sigma_M$ ***(LMM)***

***Else, there are three distinct eigenvalues (LMH).***

"MMM" is shorthand to indicate that the middle eigenvalue has multiplicity 3 and therefore all three eigenvalues are equal. MMH implies that the middle eigenvalue has multiplicity 2 and the remaining eigenvalue is larger. LMM means the middle eigenvalue has multiplicity 2 and the remaining eigenvalue is smaller. Finally, LMH means the middle eigenvalue has multiplicity 1 and therefore all three eigenvalues are distinct.†

* Though we are focusing here on *symmetric* spectral decompositions, any *diagonalizable* non-symmetric tensor admits a similar spectral decomposition except the projectors become nonsymmetric (i.e., *oblique* instead of closest-point) and are constructed from dyads of right and left eigenvectors.

† If the tensor $\underset{\approx}{\boldsymbol{A}}$ is a *stress* (positive in tension and negative in compression), then MMM is referred to as a "hydrostatic" stress state, MMH is "triaxial extension", and LMM is "triaxial compression."

Being useful in their own right, the unit tensors $\hat{\underset{\approx}{S}}$ and $\hat{\underset{\approx}{T}}$ defined in Eq. 21.122 might already be available at the time eigenspace projectors are needed. If so, the projectors may be computed directly from $\hat{\underset{\approx}{S}}$ and $\hat{\underset{\approx}{T}}$ as described below in the final summary algorithm.

***Algorithm for algebraic determination of eigenvalues and eigenspace projectors for a symmetric 3x3 tensor $\underset{\sim}{A}$.*** Let $\underset{\sim}{A}$ denote a symmetric second-order tensor having a $3\times 3$ component matrix $[A]$. The following step-by-step sequence of calculations shows how to compute the eigenvalues $\lambda_L \le \lambda_M \le \lambda_H$ and associated unique eigenprojector tensors. Individual eigenvectors, if desired, may be computed by Gram-Schmidt orthogonalization of the columns of eigenprojectors. This algorithm also shows how to efficiently compute a unit tensor $\hat{\underset{\sim}{S}}$ in the direction of the deviatoric part of $\underset{\sim}{A}$ as well as a unit tensor $\hat{\underset{\sim}{T}}$ in the direction of the deviatoric part of $\hat{\underset{\sim}{S}}^2$. The algorithm branches depending on the multiplicity of the middle eigenvalue. MMM denotes the case of triple multiplicity; LMM denotes double multiplicity with $a_L < a_M$; MMH denotes double multiplicity with $a_M < a_H$; and LMH denotes the general case of three distinct eigenvalues. The first step is to compute the following:

$$z = \frac{\mathrm{tr}\underset{\sim}{A}}{\sqrt{3}} = \frac{(A_{11}+A_{22}+A_{33})}{\sqrt{3}} \tag{21.129}$$

$$\underset{\sim}{S} = \underset{\sim}{A} - \frac{z}{\sqrt{3}}\underset{\sim}{I} \tag{21.130}$$

$$r = \left\|\underset{\sim}{S}\right\| = +\sqrt{S_{11}^2+S_{22}^2+S_{33}^2+2(S_{12}^2+S_{23}^2+S_{31}^2)}\,. \tag{21.131}$$

### SPECIAL CASE: TRIPLE MULTIPLICITY

If $r = 0$, then

---

**MMM** ($r = 0$): one distinct eigenvalue: $\lambda_L = \lambda_M = \lambda_H = \dfrac{z}{\sqrt{3}}$.

Then $\underset{\sim}{A} = \lambda_M \underset{\sim}{P}_M$, where $\underset{\sim}{P}_M = \underset{\sim}{I}$

For isotropic tensors, neither $\hat{\underset{\sim}{S}}$ nor $\hat{\underset{\sim}{T}}$ are meaningful (for the same reason that base vectors $\underset{\sim}{e}_r$ and $\underset{\sim}{e}_\theta$ lack meaning when an ordinary 3D position vector $\underset{\sim}{x}$ happens to point along the symmetry axis when using cylindrical coordinates). For MMM, the Lode angle is inconsequential, as are the tensors $\hat{\underset{\sim}{S}}$ and $\hat{\underset{\sim}{T}}$. If desired, set them to the values corresponding to pure shear. For MMM, the projectors $\underset{\sim}{P}_L$ and $\underset{\sim}{P}_H$ should be set to zero (spares) so that, for any function $f$, $f(\underset{\sim}{A}) = f(\lambda_L)\underset{\sim}{P}_L + f(\lambda_M)\underset{\sim}{P}_M + f(\lambda_H)\underset{\sim}{P}_H$. It must be understood, however, that the zero tensor is not really the projector associated with either $\lambda_L$ and $\lambda_H$. In this special MMM case, where $r = 0$ has been detected, calculations may be terminated now.

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If $r \neq 0$, then continue with calculations as follows.

$$\hat{\underset{\approx}{S}} = \frac{\underset{\approx}{S}}{r} \tag{21.132}$$

$$\sin 3\theta = 3\sqrt{6}\det\hat{\underset{\approx}{S}} = 3\sqrt{6}[\hat{S}_{11}\hat{S}_{22}\hat{S}_{33} + 2(\hat{S}_{12}\hat{S}_{23}\hat{S}_{31}) - (\hat{S}_{11}\hat{S}_{23}^2 + \hat{S}_{22}\hat{S}_{31}^2 + \hat{S}_{33}\hat{S}_{12}^2)] \tag{21.133}$$

Note: $\theta$ is called the **Lode** angle.

### SPECIAL CASE: DOUBLE MULTIPLICITY

If $\sin 3\theta = 1$, then

---

**MMH** ($r \neq 0$ and $\sin 3\theta = 1$): Two distinct eigenvalues: $\lambda_L = \lambda_M < \lambda_H$.

Then $\underset{\approx}{A} = \lambda_H \underset{\approx}{P}_H + \lambda_M \underset{\approx}{P}_M$, where $\lambda_M = \dfrac{z - r/\sqrt{2}}{\sqrt{3}}$ and $\lambda_H = \dfrac{z + \sqrt{2}r}{\sqrt{3}}$

$$\underset{\approx}{P}_H = \frac{\underset{\approx}{A} - \lambda_M \underset{\approx}{I}}{(\lambda_H - \lambda_M)} = \frac{1}{3}(\underset{\approx}{I} + \sqrt{6}\hat{\underset{\approx}{S}}) \qquad \text{and} \qquad \underset{\approx}{P}_M = \underset{\approx}{I} - \underset{\approx}{P}_H$$

For MMH, $\hat{\underset{\approx}{T}} = \hat{\underset{\approx}{S}}$. For MMH, the projector $\underset{\approx}{P}_L$ may be set to zero if you want to use an inflated spectral decomposition, and calculations may be terminated now.

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If $\sin 3\theta = -1$, then

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**LMM** ($r \neq 0$ and $\sin 3\theta = -1$): Two distinct eigenvalues: $\lambda_L < \lambda_M = \lambda_H$.

Then $\underset{\approx}{A} = \lambda_L \underset{\approx}{P}_L + \lambda_M \underset{\approx}{P}_M$, where $\lambda_M = \dfrac{z + r/\sqrt{2}}{\sqrt{3}}$ and $\lambda_L = \dfrac{z - \sqrt{2}r}{\sqrt{3}}$

$$\underset{\approx}{P}_L = \frac{\underset{\approx}{A} - \lambda_M \underset{\approx}{I}}{(\lambda_L - \lambda_M)} = \frac{1}{3}(\underset{\approx}{I} - \sqrt{6}\hat{\underset{\approx}{S}}) \qquad \text{and} \qquad \underset{\approx}{P}_M = \underset{\approx}{I} - \underset{\approx}{P}_L$$

For LMM, $\hat{\underset{\approx}{T}} = -\hat{\underset{\approx}{S}}$. For LMM, the projector $\underset{\approx}{P}_H$ may be set to zero, and calculations may be terminated now.

---

## GENERAL CASE: ALL DISTINCT EIGENVALUES (LMH)

To reach this point, there must be no repeated eigenvalues (LMH: $\lambda_L < \lambda_M < \lambda_H$), and the calculation proceeds as follows

$$\hat{\underset{\approx}{T}} = \sqrt{6}\left[\hat{\underset{\approx}{S}}^2 - \frac{1}{3}\underset{\approx}{I}\right] \tag{21.134}$$

Note, if using a rectangular Cartesian system (RCS) for components,

$$\hat{T}_{11} = \sqrt{6}\left(\hat{S}_{11}^2 + \hat{S}_{12}^2 + \hat{S}_{31}^2 - \frac{1}{3}\right) \qquad \hat{T}_{12} = \sqrt{6}(\hat{S}_{31}\hat{S}_{23} - \hat{S}_{12}\hat{S}_{33})$$

$$\hat{T}_{22} = \sqrt{6}\left(\hat{S}_{22}^2 + \hat{S}_{23}^2 + \hat{S}_{12}^2 - \frac{1}{3}\right) \qquad \hat{T}_{23} = \sqrt{6}(\hat{S}_{12}\hat{S}_{31} - \hat{S}_{23}\hat{S}_{11})$$

$$\hat{T}_{33} = \sqrt{6}\left(\hat{S}_{33}^2 + \hat{S}_{31}^2 + \hat{S}_{23}^2 - \frac{1}{3}\right) \qquad \hat{T}_{31} = \sqrt{6}(\hat{S}_{23}\hat{S}_{12} - \hat{S}_{31}\hat{S}_{22}) .$$

The eigenvalues are*

$$\lambda_H = \frac{z}{\sqrt{3}} + \frac{r}{\sqrt{2}}\left[\frac{\sin\theta}{\sqrt{3}} + \cos\theta\right] \tag{21.135a}$$

$$\lambda_M = \frac{z}{\sqrt{3}} - \sqrt{\frac{2}{3}} r \sin\theta \tag{21.135b}$$

$$\lambda_L = \frac{z}{\sqrt{3}} - \frac{r}{\sqrt{2}}\left[\frac{\sin\theta}{\sqrt{3}} - \cos\theta\right] \tag{21.135c}$$

The eigenprojectors are

$$\underset{\approx}{P}_L = \frac{(\underset{\approx}{A} - \lambda_L \underset{\approx}{I})^C}{(\lambda_M - \lambda_L)(\lambda_H - \lambda_L)}, \qquad \underset{\approx}{P}_M = \frac{(\underset{\approx}{A} - \lambda_M \underset{\approx}{I})^C}{(\lambda_H - \lambda_M)(\lambda_L - \lambda_M)}, \quad \text{and} \quad \underset{\approx}{P}_H = \frac{(\underset{\approx}{A} - \lambda_H \underset{\approx}{I})^C}{(\lambda_L - \lambda_H)(\lambda_M - \lambda_H)} \tag{21.136}$$

These formulas use the "C" cofactor operation. For any symmetric $3 \times 3$ tensor $\underset{\approx}{X}$, the RCS components of the cofactor $\underset{\approx}{X}^C$ are found by

$$X_{11}^C = X_{22}X_{33} - X_{23}^2 \qquad X_{12}^C = X_{23}X_{31} - X_{33}X_{12}$$

$$X_{22}^C = X_{33}X_{11} - X_{31}^2 \qquad X_{23}^C = X_{31}X_{12} - X_{11}X_{23}$$

$$X_{33}^C = X_{11}X_{22} - X_{12}^2 \qquad X_{31}^C = X_{12}X_{23} - X_{22}X_{31} .$$

Calculations are now complete.

Incidentally, the eigenprojectors can be computed alternatively through the following sequence of calculations:

* These formulas can be expressed in an algebraically simpler form using trigonometric identities, but this form is computationally more efficient because $s = \sin\theta$ and $c = \cos\theta$ can be computed once and saved for all subsequent calculations involving $\theta$.

$$\hat{\underset{\approx}{E}}_1 = \frac{\cos 2\theta\, \hat{\underset{\approx}{S}} - \sin\theta\, \hat{\underset{\approx}{T}}}{\cos 3\theta}, \qquad \hat{\underset{\approx}{E}}_2 = \frac{-\sin 2\theta\, \hat{\underset{\approx}{S}} + \cos\theta\, \hat{\underset{\approx}{T}}}{\cos 3\theta} \tag{21.137a}$$

$$\underset{\approx}{P}_H = \frac{1}{3}\underset{\approx}{I} + \frac{\hat{\underset{\approx}{E}}_1}{\sqrt{2}} + \frac{\hat{\underset{\approx}{E}}_2}{\sqrt{6}}, \qquad \underset{\approx}{P}_M = \frac{1}{3}\underset{\approx}{I} - 2\frac{\hat{\underset{\approx}{E}}_2}{\sqrt{6}}, \qquad \text{and} \qquad \underset{\approx}{P}_L = \frac{1}{3}\underset{\approx}{I} - \frac{\hat{\underset{\approx}{E}}_1}{\sqrt{2}} + \frac{\hat{\underset{\approx}{E}}_2}{\sqrt{6}}. \tag{21.137b}$$

As explained below, an alternative way to compute $\hat{\underset{\approx}{T}}$ proceeds as follows. First evaluate the **Lode cylindrical basis tensors**:

$$\hat{\underset{\approx}{E}}_z = \frac{\underset{\approx}{I}}{\sqrt{3}}, \qquad \hat{\underset{\approx}{E}}_r = \hat{\underset{\approx}{S}}, \qquad \hat{\underset{\approx}{E}}_\theta = \frac{\hat{\underset{\approx}{T}} - (\sin 3\theta)\hat{\underset{\approx}{S}}}{\cos 3\theta} \tag{21.138}$$

where

$$\hat{\underset{\approx}{T}} = \sqrt{6}\hat{\underset{\approx}{E}}_r^C + \frac{\hat{\underset{\approx}{E}}_z}{\sqrt{2}} \tag{21.139}$$

These three tensors, $\{\hat{\underset{\approx}{E}}_r, \hat{\underset{\approx}{E}}_\theta, \hat{\underset{\approx}{E}}_z\}$ form a basis for the 3D tensor space of all tensors that commute with $\underset{\approx}{A}$.* The isotropic part of $\underset{\approx}{A}$ is a multiple of the first base tensor $\hat{\underset{\approx}{E}}_z$. The deviatoric part of $\underset{\approx}{A}$ is a multiple of $\hat{\underset{\approx}{E}}_r$. This allows $\underset{\approx}{A}$ to be written in a familiar form for cylindrical coordinates,

$$\underset{\approx}{A} = r\hat{\underset{\approx}{E}}_r + z\hat{\underset{\approx}{E}}_z, \tag{21.140}$$

and any tensor $\underset{\approx}{C}$ that commutes with $\underset{\approx}{A}$ may be written

$$\underset{\approx}{C} = c_r\hat{\underset{\approx}{E}}_r + c_\theta\hat{\underset{\approx}{E}}_\theta + c_z\hat{\underset{\approx}{E}}_z, \tag{21.141}$$

where

$$c_r = \underset{\approx}{C}:\hat{\underset{\approx}{E}}_r \qquad c_\theta = \underset{\approx}{C}:\hat{\underset{\approx}{E}}_\theta \qquad c_z = \underset{\approx}{C}:\hat{\underset{\approx}{E}}_z, \tag{21.142}$$

which is especially convenient in isotropic plasticity theory where many of the second-order tensors (such as the normal to the yield surface) commute with stress. Moreover, the derivative of any isotropic function $\phi(\underset{\approx}{A})$ also takes the familiar form

$$\frac{d\phi}{d\underset{\approx}{A}} = \frac{\partial\phi}{\partial r}\hat{\underset{\approx}{E}}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\underset{\approx}{E}}_\theta \tag{21.143}$$

* As discussed below, the space is 3D only in the LMH case where all eigenvalues of [A] are distinct. The basis is not well-defined when there are repeated eigenvalues. Such as situation is similar to ordinary cylindrical bases in the sense that neither the radial nor angular base vectors are uniquely defined whenever the position vector resides on the symmetry axis $(r = 0)$.

***Why does this algorithm work? (further insight).*** The eigenvalues $(\lambda_L, \lambda_M, \lambda_H)$ may be regarded as Cartesian coordinates of a "pseudo-vector" in an ordinary 3D space. The prefix "pseudo" is used because the basis that goes with these coordinates is not a triad of vectors. Instead, the eigenprojector *tensors* are basis for this space. The pseudo-basis may be denoted $(\underset{\sim}{e}_L, \underset{\sim}{e}_M, \underset{\sim}{e}_H)$. Within such a space, we can always set up an alternative basis for which $\underset{\sim}{e}_3$ points along the [111] direction (i.e., $\underset{\sim}{e}_3 = \{1,1,1\}/\sqrt{3}$). The orientation of the other two base vectors is arbitrary, and we choose $\underset{\sim}{e}_1 = \{-1,0,1\}/\sqrt{2}$. Of course, this means that $\underset{\sim}{e}_2 = \underset{\sim}{e}_3 \times \underset{\sim}{e}_1 = \{1,-2,1\}/\sqrt{6}$. A point that originally had coordinates $(\lambda_L, \lambda_M, \lambda_L)$ with respect to the $(\underset{\sim}{e}_L, \underset{\sim}{e}_M, \underset{\sim}{e}_H)$ basis now has Cartesian coordinates $(x_1, x_2, x_3)$ with respect to the differently oriented basis (see figure). We may introduce a *cylindrical* basis $(\underset{\sim}{e}_r, \underset{\sim}{e}_\theta, \underset{\sim}{e}_z)$ within the differently oriented basis for which $\underset{\sim}{e}_z = \underset{\sim}{e}_3$. Therefore $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and $x_3 = z$. The cylindrical coordinates $(r, \theta, z)$ are called **Lode cylindrical coordinates**. Moreover, if $\lambda_L \le \lambda_M \le \lambda_H$, note that and the point $(\lambda_L, \lambda_M, \lambda_H)$ will always fall in the region shaded in Fig. 21.1. Consequently, the Lode angle will satisfy $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$. Conversely, if we seek ordered eigenvalues, we should always use Lode angle values in this range. Note from the figure that $\theta = \frac{\pi}{6}$ if and only if $\lambda_L = \lambda_M$, which corresponds to axisymmetry of $\underset{\sim}{A}$ with the single multiplicity eigenvalue being larger than the double root. Also, $\theta = -\frac{\pi}{6}$ if and only if $\lambda_M = \lambda_H$, which also corresponds to axisymmetry of $\underset{\sim}{A}$, but with the single root being smaller than the double root. The reference angle $\theta = 0$ corresponds to the non-axisymmetric state of pure shear in which all eigenvalues are distinct, and exactly one of the eigenvalues of the deviator is zero (making the others equal and opposite in sign). The cylindrical coordinate $r$ is zero if and only if the point $(\lambda_L, \lambda_M, \lambda_L)$ falls on the [111] symmetry axis where $\lambda_L = \lambda_M = \lambda_H$. The observation that the boundaries of allowable ranges of $\theta$ and $r$ correspond to multiple roots is the motivation for our four eigenvalue ordering subcases (MMM, LMM, MMH, LMH).

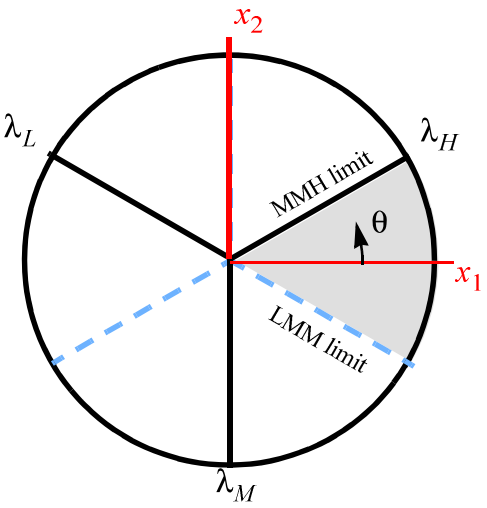


*Figure 21.1. The "pi-plane".* This is the view looking down the [111] direction. The axes to which $(\lambda_L, \lambda_M, \lambda_H)$ are referenced are the dark lines, pointing partly out of the page.

The 3D space in which $(\lambda_L, \lambda_M, \lambda_H)$ is plotted is *isomorphic** to the actual tensor subspace that we are interested in. Rather than persisting to speak of pseudo 3D base vectors, we now want to describe how the vectors in the pseudo 3D space are isomorphic (analogous) to actual tensors. The [111] axis is isomorphic to the space of isotropic tensors. The axis base vector $\underset{\sim}{\boldsymbol{e}}_3$ (or $\underset{\sim}{\boldsymbol{e}}_z$) is isomorphic to $\hat{\underset{\approx}{\boldsymbol{E}}}_3 = \underset{\approx}{\boldsymbol{I}}/\sqrt{3}$, which is simply a unit tensor in the direction of $\underset{\approx}{\boldsymbol{I}}$. The Lode radius $r$ is isomorphic to the tensor magnitude of $\underset{\approx}{\boldsymbol{S}}$ (= the deviatoric part of $\underset{\approx}{\boldsymbol{A}}$). The Lode radial base vector $\underset{\sim}{\boldsymbol{e}}_r$ is isomorphic to $\hat{\underset{\approx}{\boldsymbol{S}}}$. The base vectors $\underset{\sim}{\boldsymbol{e}}_1$ and $\underset{\sim}{\boldsymbol{e}}_2$, which have meaning only if $r \neq 0$, are isomorphic to the Lode Cartesian base tensors $\hat{\underset{\approx}{\boldsymbol{E}}}_1$ and $\hat{\underset{\approx}{\boldsymbol{E}}}_2$ computed in the LMH subcase of the above algorithm. In the LMH case, the *original* base vectors $(\underset{\sim}{\boldsymbol{e}}_L, \underset{\sim}{\boldsymbol{e}}_M, \underset{\sim}{\boldsymbol{e}}_H)$ are isomorphic to the eigenprojectors $(\underset{\approx}{\boldsymbol{P}}_L, \underset{\approx}{\boldsymbol{P}}_M, \underset{\approx}{\boldsymbol{P}}_H)$. **Symmetric commuting space** is defined to be the set of all symmetric tensors that commute with (i.e., share eigenvectors with) $\underset{\approx}{\boldsymbol{A}}$. For LMH, this commuting space is 3D. The isomorphism between ordinary 3D space and commuting tensor space is perfect as long as all three eigenvalues are distinct. When there are repeated eigenvalues, commuting space grows *larger* in dimension. For example, if $\underset{\approx}{\boldsymbol{A}}$ is isotropic, then *any* vector is an eigenvector and *any* symmetric tensor will commute with it, making commuting space equal to the entirety of symmetric tensor space and therefore *six*-dimensional (the isomorphism breaks down). If $\underset{\approx}{\boldsymbol{A}}$ has two distinct eigenvalues (one being a double root), then commuting space is larger than 3D for similar reasons.

When there are repeated eigenvalues, the concept of three Lode base tensors becomes untenable. Three independent base tensors exist only at points where lines of constant $I_1, J_2,$ and $J_3$ form a well-defined grid. As seen in Fig. 21.2, lines of constant $J_3$ are *parallel* to lines of constant $J_2$ at the LMM and MMH Lode angles. At the LMM or MMH limits, where only one eigenvalue is repeated, we can at least define the "radial" base tensor $\hat{\underset{\approx}{\boldsymbol{E}}}_r = \hat{\underset{\approx}{\boldsymbol{S}}}$. The "axial" base tensor $\hat{\underset{\approx}{\boldsymbol{E}}}_z$ can still be defined as a unit tensor in the direction of the identity tensor. In the pseudo 3D analog problem, one could easily generate the angular base vector from the radial and axial ones by $\underset{\sim}{\boldsymbol{e}}_\theta = \underset{\sim}{\boldsymbol{e}}_z \times \underset{\sim}{\boldsymbol{e}}_r$. By doing this, a basis for the 3D space can be constructed. However, in second-order *tensor* space, there is no isomorphism for the cross product.† At best, one could only assert that the "angular" base tensor $\hat{\underset{\approx}{\boldsymbol{E}}}_\theta$ would need to have a zero inner product with $\hat{\underset{\approx}{\boldsymbol{E}}}_r$ and $\hat{\underset{\approx}{\boldsymbol{E}}}_z$, implying that $\hat{\underset{\approx}{\boldsymbol{E}}}_\theta$ would belong to a 4D subspace of 6D symmetric tensor space. Symmetric commuting space is 3D only when the tensor $\underset{\approx}{\boldsymbol{A}}$ has three *distinct* eigenvalues. In that case, $\hat{\underset{\approx}{\boldsymbol{E}}}_\theta$ can be determined from $\hat{\underset{\approx}{\boldsymbol{E}}}_r$ and $\hat{\underset{\approx}{\boldsymbol{E}}}_z$, and the analog to the cross product becomes the cofactor operation indicated in Eq. (21.138). In mechanics, the LMM and MMH states correspond to axisymmetry. When eigenvalues are distinct, you might observe a material response (e.g., a localized shear band) form along a given plane. However, for axisymmetry, that same response can occur on a *family* of surfaces, all of which form equal angles with the symmetry axis. Under axisymmetric extension, for example, the failure surface observed in experiments is cone shaped — it would have to be if symmetry is to be preserved. The

* Loosely, "isomorphic" means the two systems obey the same geometry rules. A length defined in one system corresponds to the same length in the other system.

† See the brief discussion of the wedge product on page__.

variety in possible shear band orientations in axisymmetric loading is a direct result of the fact that the dimension of commuting space is larger when there are repeated eigenvalues. The material failure surface in simple shear (where eigenvalues are distinct) has no such ambiguity — it has one orientation, not a family of orientations.

LMH is the only situation for which $\hat{\boldsymbol{S}}$ and $\hat{\boldsymbol{T}}$ will be linearly independent. The tensor $\hat{\boldsymbol{S}}$ is parallel to the gradient of $J_2$, so it always points radially outward in the octahedral plane. The tensor $\hat{\boldsymbol{T}}$ is parallel to the gradient of $J_3$. As seen in Fig. 21.2, these gradient vectors are parallel in MMH and anti-parallel in LMM, making them linearly dependent in either case. The Lode Cartesian basis consists of three tensors only in this LMH case of three distinct eigenvalues. In physical applications, a supplemental tensor, $\boldsymbol{\Sigma}$, pointing in the direction of the *rate* of $\boldsymbol{A}$, is often available. If the instantaneous $\boldsymbol{A}$ tensor happens to be isotropic (three repeated roots), then the part of $\boldsymbol{\Sigma}$ perpendicular to $\boldsymbol{A}$ (namely, $\boldsymbol{\Sigma}' = \text{dev}\boldsymbol{\Sigma}$) might be nonzero, indicating that $\boldsymbol{A}$ will no longer remain isotropic an instant later in time. In this case, $\hat{\boldsymbol{S}}$ may be set equal to a unit tensor in the direction of $\boldsymbol{\Sigma}'$. Using this $\hat{\boldsymbol{S}}$, a Lode angle and associated eigenprojectors (technically, quasi-arbitrary) may be computed. For plasticity applications, $\boldsymbol{\Sigma}$ may be taken to be the trial elastic stress rate.



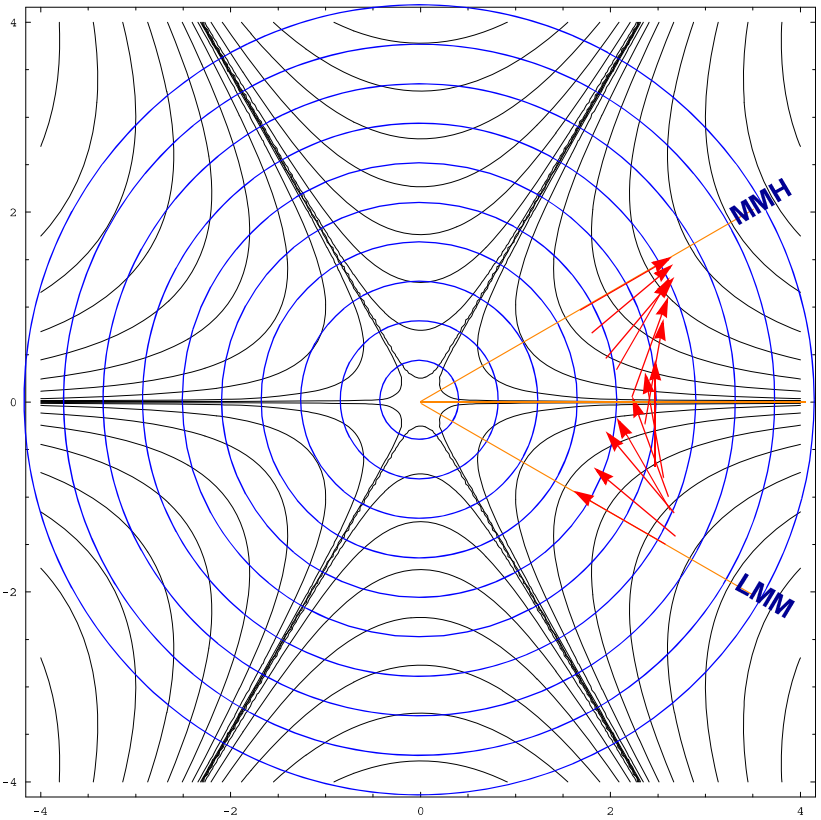


*Figure 21.2. Lines of constant $J_3$ (black) and constant $J_2$ (blue) in the pi-plane.* The arrows show the direction of the gradient of $J_3$ (i.e., the Hill tensor $\boldsymbol{T}$). Of course, the gradient of $J_2$ (i.e., the deviator $\boldsymbol{S}$) points radially outward. The Hill tensor and the deviator are linearly independent (non-parallel) everywhere except at LMM and MMH (triaxial) states.