Derivatives of principal functions

In materials modeling, researchers often use logarithmic (Hencky) strain, which is defined as the natural log of the symmetric positive-definite stretch tensor $\mathbf{U}$ from the polar decomposition of the differentiation, $\mathbf{F} = \mathbf{R} \bullet \mathbf{U}$, on page 447. Symbolically, the logarithmic strain is defined

$$\varepsilon = \ln \mathbf{U}, \tag{28.435}$$

but what does this notation mean? This strain is but one example of what we will call **principal functions**. With respect to the principal basis of $\mathbf{U}$, the component matrix for the logarithmic strain is

$$[\varepsilon] = \begin{bmatrix} \ln\lambda_1 & 0 & 0 \\ 0 & \ln\lambda_2 & 0 \\ 0 & 0 & \ln\lambda_3 \end{bmatrix} \tag{28.436}$$

where λ_1, λ_2, and λ_3 are the eigenvalues of $\mathbf{U}$ (also called "principal stretches").

Eq. (28.436) gives the component matrix *only with respect to a basis that is aligned with the principal stretch directions*. What are the components in some other basis? With respect to the laboratory basis, for example, the component matrix is

$$[\varepsilon] = [Q]\ \ln[\Lambda]\ [Q]^T \tag{28.437}$$

where the columns of $[Q]$ contain the laboratory components of the orthonormal eigenvectors of $\mathbf{U}$, and $\ln[\Lambda]$ is the diagonal matrix on the right-hand side of Eq. (28.436).

Although computing the logarithmic strain using Eq. (28.437) is reasonably straightforward, finding its *rate* is complicated considerably by the fact that not only might the stretches $[\Lambda]$ vary with time, but the stretch directions and therefore $[Q]$ might evolve as well. The rate of $[Q]$ is ambiguous because $[Q]$ itself is never unique. You can, for example, multiply any column of $[Q]$ by -1 without changing the final result of Eq. (28.437). The final result for logarithmic strain is unique *despite* non-uniqueness of $[Q]$, and it follows that (for smoothly varying stretches) the logarithmic strain *rate* must also be unique and continuous despite non-uniqueness and despite jump discontinuities in the rate of $[Q]$ at repeated eigenvalues. Rather than sticking with the special case of logarithmic strain, we will discuss how to find the rate of any principal function.

Consider any generally nonlinear tensor-to-tensor transformation (like a material model that relates stress to strain):

$$\mathbf{Y} = f(\mathbf{A}), \tag{28.438}$$

We know that

$$\dot{\mathbf{Y}} = \frac{d\mathbf{Y}}{d\mathbf{A}} : \dot{\mathbf{A}}. \tag{28.439}$$

The derivative of $\underset{\approx}{\mathbf{Y}}$ with respect to $\underset{\approx}{\mathbf{A}}$ is a fourth-order tensor having components with respect to a *fixed* orthonormal basis (e.g., the laboratory basis) given by

$$\left(\frac{d\underset{\approx}{\mathbf{Y}}}{d\underset{\approx}{\mathbf{A}}}\right)_{ijkl} = \frac{\partial Y_{ij}}{\partial A_{kl}}. \qquad (28.440)$$

But how should this derivative be computed if the basis is *not* fixed? Specifically, how does one find the rate (or other derivative) of a principal function? How, for example, does one compute the components of the stiffness tensor $\partial\sigma_{ij}/\partial\varepsilon_{kl}$ when the strain is logarithmic?

Generalizing Eq. (28.437) and assuming that $\underset{\approx}{\mathbf{A}}$ is symmetric (ensuring a diagonal component matrix in the orthonormal principal basis) the matrix version of Eq. (28.438) is

$$[\underset{\approx}{\mathbf{Y}}] = [Q]\ f[\Lambda]\ [Q]^T\ , \qquad \text{where } f[\Lambda] = \begin{bmatrix} f(\lambda_1) & 0 & 0 \\ 0 & f(\lambda_2) & 0 \\ 0 & 0 & f(\lambda_3) \end{bmatrix} \qquad (28.441)$$

The eigenvalues are now those of the tensor $\underset{\approx}{\mathbf{A}}$ and the columns of $[Q]$ contain the corresponding eigenvectors. Of course, Eq. (28.437) is merely a special case for which the function "f" is the natural logarithm.

Before proceeding with our discussion of rates, note that Eq. (28.441) may be written in a more sophisticated way as a spectral expansion. Let the "quasi-arbitrary" spectral expansion of $\underset{\approx}{\mathbf{A}}$ be given by

$$\underset{\approx}{\mathbf{A}} = \lambda_1\underset{\approx}{\mathbf{P}}_1 + \lambda_2\underset{\approx}{\mathbf{P}}_2 + \lambda_3\underset{\approx}{\mathbf{P}}_3 \qquad (28.442)$$

in which each eigenprojector $\underset{\approx}{\mathbf{P}}_k$ is the dyad of the k^{th} unit eigenvector $\underset{\sim}{\mathbf{p}}_k$ with itself. The expansion is called "quasi-arbitrary" because each $\underset{\approx}{\mathbf{P}}_k$ is unique only if its associated eigenvalue has multiplicity one — eigenvectors for double or triple roots are non-unique in orientation, making their associated eigenprojectors non-unique as well. Despite this ambiguity, the expansion of Eq. (28.442) produces a unique result, just as the final result of Eq. (28.441) is unique despite non-uniqueness of $[Q]$. Given the spectral expansion of Eq. (28.442), $\underset{\approx}{\mathbf{Y}} = f(\underset{\approx}{\mathbf{A}})$ is simply

$$f(\underset{\approx}{\mathbf{A}}) = f(\lambda_1)\underset{\approx}{\mathbf{P}}_1 + f(\lambda_2)\underset{\approx}{\mathbf{P}}_2 + f(\lambda_3)\underset{\approx}{\mathbf{P}}_3 \qquad (28.443)$$

For problems in which $\underset{\approx}{\mathbf{A}}$ varies in time*, not only will the eigenvalues of $\underset{\approx}{\mathbf{A}}$ vary, but so will the eigenvectors. The $[Q]$ matrix and, equivalently, the $\underset{\approx}{\mathbf{P}}_k$ tensors have rates associated with evolving orientations for the principal basis, and they might even suffer jump discontinuities when eigenvalues are not distinct. We are now poised to explain that a unique and well-behaved rate of $\underset{\approx}{\mathbf{Y}} = f(\underset{\approx}{\mathbf{A}})$ exists despite these vexing complications. In what follows, we will no longer presume that $\underset{\approx}{\mathbf{A}}$ is symmetric. Instead we suppose only that it is diagonalizable. When reading the upcoming *ugly* formulas, it might help to look at the example on page 759.

* We will refer to t as "time," but really it can represent any scalar on which $\underset{\approx}{\mathbf{A}}$ depends.

TENSOR CALCULUS

PROBLEM STATEMENT

If a diagonalizable tensor $\underset{\approx}{A}$ has a quasi-arbitrary spectral expansion $\underset{\approx}{A} = \sum_{i=1}^{3} \lambda_i \underset{\approx}{P}_i$,

and if a second tensor is a principal function of $\underset{\approx}{A}$, defined by $\underset{\approx}{Y} = \sum_{i=1}^{3} f(\lambda_i) \underset{\approx}{P}_i$, what

is the time rate $\dot{\underset{\approx}{Y}}$? More generally, what is $\dfrac{d\underset{\approx}{Y}}{d\underset{\approx}{A}}$?

Here, $\{\lambda_1, \lambda_2, \lambda_3\}$ are the eigenvalues of $\underset{\approx}{A}$ and $\{\underset{\approx}{P}_1, \underset{\approx}{P}_2, \underset{\approx}{P}_3\}$ are the associated quasi-arbitrary eigenprojectors. In conventional matrix analysis parlance, this problem statement would read as follows: If $[A] = [L][\Lambda][L]^{-1}$ (see Eq. 7.16), where $[\Lambda]$ is the diagonal matrix of eigenvalues and $[L]$ is the matrix containing associated eigenvectors in its columns (when [A] is symmetric, [L] is the direction cosine matrix [Q], and $[L]^{-1}$ is just $[Q]^T$), and if $[Y] = [L][f(\Lambda)][L]^{-1}$, then what is $[\dot{Y}]$? The answer must include contributions from both $\dot{\Lambda}$ and $\dot{L}$, so this is a very difficult problem. Let's now return to direct structured notation.

For a diagonalizable tensor $\underset{\approx}{A}$, the geometric multiplicity of each eigenvalue equals its algebraic multiplicity. In other words, there will always exist three linearly independent eigenvectors $\{\underset{\sim}{p}_1, \underset{\sim}{p}_2, \underset{\sim}{p}_3\}$ associated with the eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$. Eigenvectors are, of course, always arbitrary in magnitude. When there are repeated eigenvalues, they are also quasi-arbitrary in direction. We will limit attention to the case where the eigenvalues (and therefore the eigenvectors) are real.* Each quasi-arbitrary eigenprojector $\underset{\approx}{P}_i$ is formed from the eigenvector $\underset{\sim}{p}_k$ by the dyadic product $\underset{\sim}{p}_k \underset{\sim}{p}^k$ (no sum on k), where $\underset{\sim}{p}^k$ is the k^{th} *dual* vector associated with the triad $\overline{\{\underset{\sim}{p}_1, \underset{\sim}{p}_2, \underset{\sim}{p}_3\}}$, and the overbar denotes the complex conjugate. If $\underset{\approx}{A}$ is symmetric, then the triad $\{\underset{\sim}{p}_1, \underset{\sim}{p}_2, \underset{\sim}{p}_3\}$ is real and may be presumed *orthonormal*, in which case there is no distinction between $\underset{\sim}{p}^k$ and $\underset{\sim}{p}_k$.

If $\underset{\approx}{A}$ is non-symmetric and diagonalizable (but with real eigenvalues), then the dual (left) eigenvectors $\{\underset{\sim}{p}^1, \underset{\sim}{p}^2, \underset{\sim}{p}^3\}$ are determined from the (right) eigenvectors $\{\underset{\sim}{p}_1, \underset{\sim}{p}_2, \underset{\sim}{p}_3\}$ by first computing a 3×3 matrix $[g]$ whose components are given by

$$g_{ij} \equiv \underset{\sim}{p}_i \bullet \underset{\sim}{p}_j . \qquad (28.444)$$

Letting g^{ij} denote the components of the matrix $[g]^{-1}$, the dual vectors are then found by

* The concepts in this section *can* be generalized to the complex case, but we wish to present the material in the simpler context of real vectors and tensors.

$$\tilde{p}^i = \sum_{j=1}^{3} g^{ij} \tilde{p}_j . \tag{28.445}$$

Alternatively, if $[L]$ is the matrix whose columns contain the lab components of the right eigenvectors $\{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$, then the *rows* of $[L]^{-1}$ will contain the lab components of the dual (left) eigenvectors $\{\tilde{p}^1, \tilde{p}^2, \tilde{p}^3\}$.

Now that we have stated the problem, what is the answer? The formula for the rate of a principal function, $\mathbf{Y} = f(\mathbf{A})$, may be written entirely in direct notation as follows:

If $\mathbf{Y} = \sum_{i=1}^{3} f(\lambda_i)\mathbf{P}_i$, where $\{\lambda_1, \lambda_2, \lambda_3\}$ are the eigenvalues of some symmetric tensor $\mathbf{A}$ and $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ are the associated eigen*projectors*, then

$$\dot{\mathbf{Y}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \gamma_{ij} \mathbf{P}_i \bullet \dot{\mathbf{A}} \bullet \mathbf{P}_j \quad \textbf{where} \quad \gamma_{ij} = \begin{cases} \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} & \text{if} \quad \lambda_i \neq \lambda_j \\ f'(\lambda_i) & \text{if} \quad \lambda_i = \lambda_j \end{cases}$$

If follows that

$$\frac{d\mathbf{Y}}{d\mathbf{A}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \gamma_{ij} (\mathbf{P}_i \mathbf{P}_j)^L \quad \text{where} \quad \gamma_{ij} = \begin{cases} \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} & \text{if} \quad \lambda_i \neq \lambda_j \\ f'(\lambda_i) & \text{if} \quad \lambda_i = \lambda_j \end{cases} \tag{28.446}$$

Where the superscript "L" denotes the leafing operation defined in Eq. (18.52)* on page 344. When the tensor $\mathbf{A}$ is symmetric (both at this instant and at all other times, as it is for logarithmic strain), then the tangent derivative should be converted to a *limited dimension* derivative as

$$\frac{d\mathbf{Y}}{d\mathbf{A}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \gamma_{ij} (\mathbf{P}_i \mathbf{P}_j)^{\Lambda} \quad \text{where} \quad \gamma_{ij} = \begin{cases} \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} & \text{if} \quad \lambda_i \neq \lambda_j \\ f'(\lambda_i) & \text{if} \quad \lambda_i = \lambda_j \end{cases} \tag{28.447}$$

where the superscript "Λ" denotes the symmetric leafing operation defined in Eq. (18.63) on page 346.† In this case of a symmetric argument, the Mandel component matrix for the rate $\mathbf{Y}$ in terms of the principal basis may be computed by the matrix multiplication

* If H_{ijkl} are components of a fourth-order tensor, then the components of its "leaf" are shuffled like cards; namely H_{ikjl}. This index re-ordering is similar in spirit to index reordering in a simple transpose operation.

† The symmetric leaf of H_{ijkl} is a new tensor with components $\frac{1}{4}(H_{ikjl} + H_{kijl} + H_{iklj} + H_{kilj})$.

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$$\begin{bmatrix} (\dot{\underset{\approx}{Y}})^{<A>}_{11} \\ (\dot{\underset{\approx}{Y}})^{<A>}_{22} \\ (\dot{\underset{\approx}{Y}})^{<A>}_{33} \\ (\dot{\underset{\approx}{Y}})^{<A>}_{23} \\ (\dot{\underset{\approx}{Y}})^{<A>}_{31} \\ (\dot{\underset{\approx}{Y}})^{<A>}_{12} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_{12} \end{bmatrix} \begin{bmatrix} (\dot{\underset{\approx}{A}})^{<A>}_{11} \\ (\dot{\underset{\approx}{A}})^{<A>}_{22} \\ (\dot{\underset{\approx}{A}})^{<A>}_{33} \\ (\dot{\underset{\approx}{A}})^{<A>}_{23} \\ (\dot{\underset{\approx}{A}})^{<A>}_{31} \\ (\dot{\underset{\approx}{A}})^{<A>}_{12} \end{bmatrix}. \tag{28.448}$$

The superscript “<A>” indicates that the components are referenced to the principal directions of $\underset{\approx}{A}$. The 6×6 matrix in this equation is the principal Mandel matrix for $d\underset{\approx}{Y}/d\underset{\approx}{A}$. See page 761 for an example. For numerical implementation, this result may be computed directly from the *laboratory* components of $\underset{\approx}{A}$ and $\underset{\approx}{Y}$ by

$$\dot{Y}^{<lab>}_{mn} = \sum_{r=1}^{3}\sum_{s=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(Q_{mk}Q_{nl}Q_{rk}Q_{sl}\gamma_{kl})\dot{A}^{<lab>}_{pq} \quad \textbf{where} \quad \gamma_{ij} = \begin{cases} \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} & \text{if} \quad \lambda_i \neq \lambda_j \\ f'(\lambda_i) & \text{if} \quad \lambda_i = \lambda_j \end{cases}$$

Here, Q_{ij} denotes the i^{th} lab component of the j^{th} *orthonormalized* eigenvector of $\underset{\approx}{A}$ (i.e., $Q_{ij} \equiv \underset{\sim}{E}_i \bullet \underset{\sim}{p}_j$ so that the columns of $[Q]$ contain the lab components of the eigenvectors).

This result holds even if the tensor $\underset{\approx}{A}$ has repeated eigenvalues — this formula still gives the correct answer no matter what choice is made for the orthonormalized eigenvectors despite the fact they are not unique when there are repeated eigenvalues!

So far, we taken the $\underset{\approx}{P}_k$ tensors to be the *quasi-arbitrary* eigenprojectors formed by the dyad of each eigenvector ($\underset{\approx}{P}_k = \underset{\sim}{p}_k \underset{\sim}{p}^k$, with no sum on *k*). When an eigenvalue is a double or triple root, these projectors are not unique, but the formulas still give the correct result in the end.

Computation of the derivative may be greatly simplified by using the unique *primary* eigenprojectors associated with each distinct eigenvalue λ_k. In this case, if there are m distinct eigenvalues, then

$$\frac{d\underset{\approx}{Y}}{d\underset{\approx}{A}} = \sum_{i=1}^{m}\sum_{j=1}^{m}\gamma_{ij}(\underset{\approx}{P}_i\underset{\approx}{P}_j)^{\Lambda} \quad \text{where} \quad \gamma_{ij} = \begin{cases} \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} & \text{if} \quad \lambda_i \neq \lambda_j \\ f'(\lambda_i) & \text{if} \quad \lambda_i = \lambda_j \end{cases}. \tag{28.449}$$

Suppose, for example, the tensor $\underset{\approx}{A}$ is isotropic at a given instant in time (there is no requirement that it *remain* isotropic). Then it is of the form $\underset{\approx}{A} = \lambda\underset{\approx}{I}$. In other words, it has only one distinct eigenvalue $(\lambda_1 = \lambda_2 = \lambda_3 = \lambda)$. In this case, all nine values of γ_{ij} simply equal $f'(\lambda)$, and Eq. (28.449) reduces to

$$\frac{d\underset{\approx}{Y}}{d\underset{\approx}{A}} = f'(\lambda)(\underset{\approx}{I}\underset{\approx}{I})^{\Lambda} \qquad \text{(this applies when } \underset{\approx}{A} = \lambda\underset{\approx}{I}\text{)} \tag{28.450}$$

where $(\underset{\approx}{\boldsymbol{I}}\underset{\approx}{\boldsymbol{I}})^{\Lambda}$ (denoted $\underset{\approx}{\boldsymbol{P}}^{\text{sym}}$ elsewhere in this book) is the fourth-order identity tensor for symmetric second-order tensors. When $\underset{\approx}{\boldsymbol{A}}$ happens to be isotropic at a given instant of time, that does *not* mean that its rate will be isotropic. The rate can be a fully populated tensor. When $\underset{\approx}{\boldsymbol{A}}$ is isotropic, the lab components $\dot{\underset{\approx}{\boldsymbol{Y}}}$ are related to the lab components of $\dot{\underset{\approx}{\boldsymbol{A}}}$ by

$$\begin{bmatrix} (\dot{\underset{\sim}{\boldsymbol{Y}}})_{11} & (\dot{\underset{\sim}{\boldsymbol{Y}}})_{12} & (\dot{\underset{\sim}{\boldsymbol{Y}}})_{13} \\ & (\dot{\underset{\sim}{\boldsymbol{Y}}})_{22} & (\dot{\underset{\sim}{\boldsymbol{Y}}})_{23} \\ \text{sym} & & (\dot{\underset{\sim}{\boldsymbol{Y}}})_{33} \end{bmatrix} = f'(\Lambda) \begin{bmatrix} (\dot{\underset{\sim}{\boldsymbol{A}}})_{11} & (\dot{\underset{\sim}{\boldsymbol{A}}})_{12} & (\dot{\underset{\sim}{\boldsymbol{A}}})_{13} \\ & (\dot{\underset{\sim}{\boldsymbol{A}}})_{22} & (\dot{\underset{\sim}{\boldsymbol{A}}})_{23} \\ \text{sym} & & (\dot{\underset{\sim}{\boldsymbol{A}}})_{33} \end{bmatrix}. \qquad \text{(this applies when } \underset{\sim}{\boldsymbol{A}} = \lambda \underset{\sim}{\boldsymbol{I}}\text{)} \qquad (28.451)$$

How do things change when the tensor $\underset{\approx}{\boldsymbol{A}}$ has exactly *two* distinct eigenvalues? If the tensor $\underset{\approx}{\boldsymbol{A}}$ is transversely isotropic, making its primary spectral expansion $\underset{\approx}{\boldsymbol{A}} = \lambda_a \underset{\approx}{\boldsymbol{P}}_a + \lambda_b \underset{\approx}{\boldsymbol{P}}_b$, then Eq. (28.449) reduces to

$$\frac{d\underset{\sim}{\boldsymbol{Y}}}{d\underset{\sim}{\boldsymbol{A}}} = f'(\lambda_a)(\underset{\sim}{\boldsymbol{P}}_a \underset{\sim}{\boldsymbol{P}}_a)^{\Lambda} + f'(\lambda_b)(\underset{\sim}{\boldsymbol{P}}_b \underset{\sim}{\boldsymbol{P}}_b)^{\Lambda} + \frac{f(\lambda_a) - f(\lambda_b)}{\lambda_a - \lambda_b}(\underset{\sim}{\boldsymbol{P}}_a \underset{\sim}{\boldsymbol{P}}_b + \underset{\sim}{\boldsymbol{P}}_a \underset{\sim}{\boldsymbol{P}}_b)^{\Lambda}, \qquad (28.452)$$

and

$$\dot{\underset{\sim}{\boldsymbol{Y}}} = f'(\lambda_a)(\underset{\sim}{\boldsymbol{P}}_a \bullet \dot{\underset{\sim}{\boldsymbol{A}}} \bullet \underset{\sim}{\boldsymbol{P}}_a) + f'(\lambda_b)\underset{\sim}{\boldsymbol{P}}_b \bullet \dot{\underset{\sim}{\boldsymbol{A}}} \bullet \underset{\sim}{\boldsymbol{P}}_b + \frac{f(\lambda_a) - f(\lambda_b)}{\lambda_a - \lambda_b}(\underset{\sim}{\boldsymbol{P}}_a \bullet \dot{\underset{\sim}{\boldsymbol{A}}} \bullet \underset{\sim}{\boldsymbol{P}}_b + \underset{\sim}{\boldsymbol{P}}_b \bullet \dot{\underset{\sim}{\boldsymbol{A}}} \bullet \underset{\sim}{\boldsymbol{P}}_a). \qquad (28.453)$$

See page 762 for an example of how to use this result.

$$\int_0^3 [D]dt \neq 0 . \tag{35.48}$$

For this path, $[F]$ has the same value at times 0 and 3. Therefore, for any matrix $[X]$ given by a function of $[F]$, the following must be true:

$$\int_0^3 [\dot{X}]dt = [X]_{\text{at time 3}} - [X]_{\text{at time 0}} = [0] . \tag{35.49}$$

In other words, if $[X]$ is function of $[F]$, then it must also have the same value at times 0 and 3 and therefore the time integral of its rate must be zero *for this problem*. Comparing Eqs. (35.49) and (35.48) proves that $[D]$ can not, in general, be expressed in the form $[\dot{X}]$ for *any* $[X]$ that is function of $[F]$. In other words, $[D]$ is not a "true rate". The time integral of $[D]$ cannot be determined solely from knowledge of the end-states — it depends on the path.

(d) This example can be interpreted in the context of continuum mechanics, where $[F]$ represents a material deformation gradient, and $[D]$ represents the symmetric part of the velocity gradient, which is often used in materials modeling *as if it were* a strain rate. This example is a *counter*-example proving that $[D]$ is *not* a true rate, so $[D]$ cannot be properly regarded as a strain rate. Also, this example uses a time-varying deformation gradient that is always symmetric and positive definite. In other words, the deformation is, at all times, a pure stretch. Nevertheless, as illustrated in this example, the velocity gradient [L] is not always symmetric.

Time rate of a principal function

This section provides an example for the theory of time rates of principal functions, discussed on page 628. Suppose that the time varying *lab* components of $\underset{\approx}{A}$ are given by

$$[A]_{\langle\text{lab}\rangle} = \begin{bmatrix} 41e^{t-1} & -12t & 0 \\ -12t & 34t^3 & \ln t \\ 0 & \ln t & 50t^2 \end{bmatrix}_{\langle\text{lab}\rangle} . \tag{35.50}$$

Differentiating the this matrix gives the *lab* components of $\dot{\underset{\approx}{A}}$:

$$[\dot{A}]_{\langle\text{lab}\rangle} = \begin{bmatrix} 41e^{t-1} & -12 & 0 \\ -12 & 102t^2 & \frac{1}{t} \\ 0 & \frac{1}{t} & 100t \end{bmatrix}_{\langle\text{lab}\rangle} . \tag{35.51}$$

For illustration, let's suppose that we wish to compute the rate of $\underset{\approx}{Y} = \underset{\approx}{A}^2$, evaluated at $t=1$. In this particular case, we know that the answer can be computed without using the specialized formulas presented on page 628. We know the answer is simply

$$\dot{\underset{\approx}{Y}} = \underset{\approx}{A} \bullet \dot{\underset{\approx}{A}} + \dot{\underset{\approx}{A}} \bullet \underset{\approx}{A} . \tag{35.52}$$

At $t=1$, Eqs. (35.50) and (35.51) evaluate to

$$[A]_{\text{<lab>}} = \begin{bmatrix} 41 & -12 & 0 \\ -12 & 34 & 0 \\ 0 & 0 & 50 \end{bmatrix}_{\text{<lab>}} \quad \text{and} \quad [\dot{A}]_{\text{<lab>}} = \begin{bmatrix} 41 & -12 & 0 \\ -12 & 102 & 1 \\ 0 & 1 & 100 \end{bmatrix}_{\text{<lab>}}, \tag{35.53}$$

from which the traditional method of Eq. (35.52) gives the rate at this instant:

$$[\dot{Y}] = \begin{bmatrix} 41 & -12 & 0 \\ -12 & 34 & 0 \\ 0 & 0 & 50 \end{bmatrix}\begin{bmatrix} 41 & -12 & 0 \\ -12 & 102 & 1 \\ 0 & 1 & 100 \end{bmatrix} + \begin{bmatrix} 41 & -12 & 0 \\ -12 & 102 & 1 \\ 0 & 1 & 100 \end{bmatrix}\begin{bmatrix} 41 & -12 & 0 \\ -12 & 34 & 0 \\ 0 & 0 & 50 \end{bmatrix} = \begin{bmatrix} 3650 & -2616 & -12 \\ -2616 & 7224 & 84 \\ -12 & 84 & 10000 \end{bmatrix}. \tag{35.54}$$

Now we wish to demonstrate that the methods cited on page 628 (and subsequent pages) will give *this same result*. At the instant of interest, the eigenvalues of the laboratory matrix $[A]_{\text{<lab>}}$ are $\{25, 50, 50\}$. Corresponding orthonormal eigenvectors are assembled into columns to create a "Q" direction cosine matrix:

$$[Q] = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 4/5 & 0 & 3/5 \\ 0 & 1 & 0 \end{bmatrix}. \tag{35.55}$$

For this example, $f(x) = x^2$. Therefore the γ_{ij} coefficients defined on page 632 are given by

$$\gamma_{ij} = \begin{cases} \dfrac{\lambda_i^2 - \lambda_j^2}{\lambda_i - \lambda_j} & \text{if} \quad \lambda_i \neq \lambda_j \\ 2\lambda_i & \text{if} \quad \lambda_i = \lambda_j \end{cases}, \qquad \text{or simply} \qquad \gamma_{ij} = \lambda_i + \lambda_j. \tag{35.56}$$

Sum convention is suspended here; *i* and *j* are *free* indices.

For this example, the function γ_{ij} reduced to a form that no longer required special-case handling when $\lambda_i = \lambda_j$. This happened because we chose a simple quadratic function $f(\lambda)$ for our example. The special-case handling will be needed only for functions, such as the logarithm, that have an infinite Taylor series.

Recalling that the eigenvalues are $\{25, 50, 50\}$, the matrix of γ_{ij} coefficients* is

$$[\gamma_{ij}] = [\lambda_i + \lambda_j] = \begin{bmatrix} 50 & 75 & 75 \\ 75 & 100 & 100 \\ 75 & 100 & 100 \end{bmatrix}. \tag{35.57}$$

The components of $\dot{\mathbf{A}}$ with respect to the *principal* basis are

$$[\dot{A}]_{\text{<A>}} = [Q]^T [\dot{A}]_{\text{<lab>}} [Q] = \begin{bmatrix} 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \\ -4/5 & 3/5 & 0 \end{bmatrix}\begin{bmatrix} 41 & -12 & 0 \\ -12 & 34 & 0 \\ 0 & 0 & 50 \end{bmatrix}\begin{bmatrix} 3/5 & 0 & -4/5 \\ 4/5 & 0 & 3/5 \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 1713 & 20 & 816 \\ 20 & 2500 & 15 \\ 816 & 15 & 1862 \end{bmatrix}_{\text{<A>}} \tag{35.58}$$

Multiplying each component of $[\dot{A}]_{\text{<A>}}$ by the corresponding component of $[\gamma_{ij}]$ gives the components of $\dot{\mathbf{Y}}$ with respect to the principal basis:

* This is just a matrix of numbers, not components of a tensor.

EXAMPLES/PRACTICE

$$[\dot{Y}]_{<A>} = \begin{bmatrix} 3426 & 60 & 2448 \\ 60 & 10000 & 60 \\ 2448 & 60 & 7448 \end{bmatrix}. \tag{35.59}$$

Transforming this result back to the lab basis gives the lab components of $\dot{\underset{\approx}{Y}}$:

$$[\dot{Y}]_{<lab>} = [Q][\dot{Y}]_{<A>}[Q]^T = \begin{bmatrix} 3/5 & 0 & -4/5 \\ 4/5 & 0 & 3/5 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3426 & 60 & 2448 \\ 60 & 10000 & 60 \\ 2448 & 60 & 7448 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ 0 & 0 & 1 \\ -4/5 & 3/5 & 0 \end{bmatrix}, \tag{35.60}$$

or

$$[\dot{Y}]_{<lab>} = \begin{bmatrix} 3650 & -2616 & -12 \\ -2616 & 7224 & 84 \\ -12 & 84 & 10000 \end{bmatrix}_{<lab>}, \tag{35.61}$$

which agrees with the result derived in the traditional manner [Eq. (35.54)].

For this particular example, the second and third eigenvalue both equaled 50. Consequently, the eigenvectors associated with this repeated eigenvalue were not unique. Nonetheless, no matter what orthonormal triad you use to construct the direction cosine matrix $[Q]$, the final result will be the same. Try it!

Let's now solve this problem using Eq. (28.453), which is cast in terms of the primary eigenspace projectors, which *are* unique. The eigenspace projector associated with the single-root eigenvalue $\lambda_a = 25$ is formed by the dyad (outer product) of the associated normalized eigenvector

$$[P]_a = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{35.62}$$

We could construct the eigenspace projector associated with the repeat-root eigenvalue $\lambda_b = 50$ by summing the dyads of its individual eigenvectors, but a faster way is to recognize that $[P]_b = [I] - [P]_a$, or

$$[P]_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{25} \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 16 & -12 & 0 \\ -12 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}. \tag{35.63}$$

In anticipation of applying Eq. (28.453), recall that $f(x) = x^2$. Therefore $f'(x) = 2x$ and

$$f(\lambda_a) = f(25) = 25^2, \qquad f(\lambda_b) = f(50) = 50^2 = (4)25^2 \tag{35.64}$$

$$f'(\lambda_a) = 2(25), \qquad f'(\lambda_b) = 2(50) = 4(25). \tag{35.65}$$

Using Eq. (35.53) for the *laboratory* components of $[\dot{A}]$, note that

$$[\underset{\sim}{P}_a \bullet \dot{\underset{\sim}{A}} \bullet \underset{\sim}{P}_a] = \frac{1}{25^2} \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 41 & -12 & 0 \\ -12 & 102 & 1 \\ 0 & 1 & 100 \end{bmatrix} \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{25^2} \begin{bmatrix} 15417 & 20556 & 0 \\ 20556 & 27408 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{35.66}$$

$$[\underset{\sim}{P}_b \bullet \dot{\underset{\sim}{A}} \bullet \underset{\sim}{P}_b] = \frac{1}{25^2}\begin{bmatrix} 16 & -12 & 0 \\ -12 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}\begin{bmatrix} 41 & -12 & 0 \\ -12 & 102 & 1 \\ 0 & 1 & 100 \end{bmatrix}\begin{bmatrix} 16 & -12 & 0 \\ -12 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix} = \frac{1}{25^2}\begin{bmatrix} 29792 & -22344 & -300 \\ -22344 & 16758 & 225 \\ -300 & 225 & 62500 \end{bmatrix} \quad (35.67)$$

$$[\underset{\sim}{P}_a \bullet \dot{\underset{\sim}{A}} \bullet \underset{\sim}{P}_b] = \frac{1}{25^2}\begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 41 & -12 & 0 \\ -12 & 102 & 1 \\ 0 & 1 & 100 \end{bmatrix}\begin{bmatrix} 16 & -12 & 0 \\ -12 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix} = \frac{1}{25^2}\begin{bmatrix} -9792 & 7344 & 300 \\ -13056 & 9792 & 400 \\ 0 & 0 & 0 \end{bmatrix} \quad (35.68)$$

$$[\underset{\sim}{P}_b \bullet \dot{\underset{\sim}{A}} \bullet \underset{\sim}{P}_a] = \frac{1}{25^2}\begin{bmatrix} 16 & -12 & 0 \\ -12 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}\begin{bmatrix} 41 & -12 & 0 \\ -12 & 102 & 1 \\ 0 & 1 & 100 \end{bmatrix}\begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{25^2}\begin{bmatrix} -9792 & -13056 & 0 \\ 7344 & 9792 & 0 \\ 300 & 400 & 0 \end{bmatrix}. \quad (35.69)$$

Recall Eq. (28.453):

$$\dot{\underset{\sim}{Y}} = f'(\lambda_a)(\underset{\sim}{P}_a \bullet \dot{\underset{\sim}{A}} \bullet \underset{\sim}{P}_a) + f'(\lambda_b)(\underset{\sim}{P}_b \bullet \dot{\underset{\sim}{A}} \bullet \underset{\sim}{P}_b) + \frac{f(\lambda_a) - f(\lambda_b)}{\lambda_a - \lambda_b}(\underset{\sim}{P}_a \bullet \dot{\underset{\sim}{A}} \bullet \underset{\sim}{P}_b + \underset{\sim}{P}_b \bullet \dot{\underset{\sim}{A}} \bullet \underset{\sim}{P}_a) \quad (35.70)$$

Substituting the above results into this equation gives

$$[\dot{\underset{\approx}{Y}}]_{\text{<lab>}} = \begin{bmatrix} 3650 & -2616 & -12 \\ -2616 & 7224 & 84 \\ -12 & 84 & 10000 \end{bmatrix}, \quad (35.71)$$

which agrees with Eqs. (35.54) and (35.61).

EXAMPLES/PRACTICE