Primer[1] on the simplest inelastic model: non-Hardening von Mises (J_2) plasticity

Several elementary tutorials on plasticity theory are available by typing “plasticity” into the search box at [the CSM website, http://csm.mech.utah.edu](http://csm.mech.utah.edu). Of these, the most complete tutorial is

Publication: Elements of Phenomenological Plasticity: Geometrical Insight, Computational Algorithms, and Topics in Shock Physics

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This 2007 Book Chapter on the basics of plasticity theory reviews the terminology and governing equations of plasticity, with emphasis on amending misconceptions, providing physical insights, and outlining computational algorithms....

The textbook,

> A. Anandarajah, Computational Methods in Elasticity and Plasticity: Solids and Porous Media, Springer | 2010 | ISBN: 1441963782 | 685 pages | $119.

has a good introduction to computational plasticity theory, with the added advantage of providing guidance on how to incorporate such models into a finite element computational framework.

Below (in this document) is a primer on simple non-hardening von Mises plasticity, serving as a starting point for reading the above resources. This primer provides background on the theory, an algorithm (with pseudo-code) and two verification test problems that should ALWAYS be run whenever you use the von Mises model in a code. For simplicity, the theory is described here for the case of small displacement gradients (which means infinitesimal strains and infinitesimal rotations).

The von Mises theory is often called “J_2 plasticity” because it is usually described in terms of the so-called second mechanics invariant of the stress,

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[1] When used to mean “tutorial,” the word “primer” is supposed to rhyme with “glimmer” not “timer.”

$$J_2 = \tfrac{1}{2} s_{ij} s_{ij} = \tfrac{1}{2} \mathrm{tr}\left(\underset{\approx}{\mathbf{s}} \cdot \underset{\approx}{\mathbf{s}} \right) = \tfrac{1}{2} \left\| \underset{\approx}{\mathbf{s}} \right\|^2 ,$$

where ,

$$\underset{\approx}{\mathbf{s}} \equiv \underset{\approx}{\boldsymbol{\sigma}} - \tfrac{1}{3}\left(\mathrm{tr}\underset{\approx}{\boldsymbol{\sigma}} \right) \underset{\approx}{\mathbf{I}}$$

Here, $\underset{\approx}{\mathbf{s}}$ is the deviatoric part of the stress tensor $\underset{\approx}{\boldsymbol{\sigma}}$. When expressed in terms of the principal stresses,

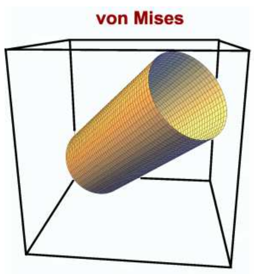
$$J_2 = \tfrac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]$$

The von Mises yield criterion states that a stress tensor is “at yield” or “on the yield surface” if

$$J_2 = k^2 ,$$

where k is a positive material constant. The material is treated as elastic or “inside the yield surface” if

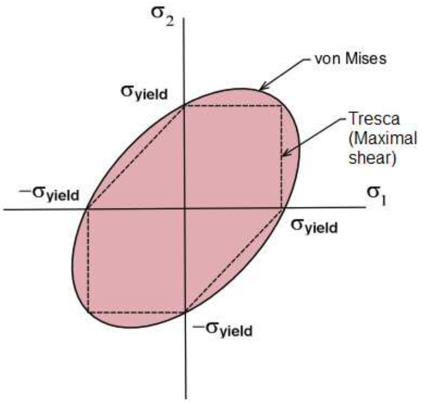
$$J_2 < k^2$$



The “yield surface” is defined to be the set of all stress states satisfying the yield criterion, $J_2 = k^2$. Substituting the previous expression for J_2 in terms of principal stresses, the yield surface is therefore defined by the set of all $\{\sigma_1, \sigma_2, \sigma_3\}$ for which

$$\tfrac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] = k^2$$

Though not immediately obvious, this describes a cylinder of radius $k\sqrt{2}$ in principal stress space. The cylinder axis is aligned with the [1,1,1] direction. In other words, it points along the diagonal of the reference box in the sketch (whose edges are the principal stress axes). The [1,1,1] direction is called the “hydrostat” because it is where all principal stresses are equal.



For the important special case of plane stress $(\sigma_3 = 0)$, this reduces to

$$\tfrac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2 \right] = k^2$$

which should come as no surprise since the intersection of the cylinder with the $\sigma_3 = 0$ plane (i.e. the back surface of the box in the 3D drawing) is a tilted ellipse.

The value of the material parameter k can be found by running one of the following two experiments:

1. Simple shear: At yield, the applied shear stress is denoted τ_y. Put the principal stresses to their values in simple shear (at yield),

$$\sigma_1 = \tau_y$$
$$\sigma_2 = -\tau_y$$
$$\sigma_3 = 0$$

Then the yield criterion reduces to

$$\tau_y^2 = k^2,$$

implying that $k = \tau_y$.

2. Uniaxial stress: At yield, the applied axial stress is denoted Y (labeled σ_{yield} in the previous tilted ellipse sketch). Put the principal stresses to their values in uniaxial stress (at yield),

$$\sigma_1 = Y$$
$$\sigma_2 = 0$$
$$\sigma_3 = 0$$

Then the yield criterion reduces to

$$\tfrac{1}{3}Y^2 = k^2,$$

implying that $k = Y/\sqrt{3}$.

When someone tells you the "yield stress," you need to ask them whether they mean yield in shear or yield in uniaxial stress so that you can assign the correct value to k. If they give you yield in uniaxial stress, *Y*, then you need to find out if the code requires yield in shear. If so, then you need to set $k = \tau_y = Y/\sqrt{3}$. The so-called Hugoniot elastic limit, σ^{HEL}, is yet another definition of yield stress defined below in the discussion of uniaxial strain loading.

WARNING: the relationship, $\tau_y = Y/\sqrt{3}$, relating yield in shear to yield in uniaxial stress does not apply in general – it is a direct consequence of the von Mises criterion. The relationship (if any) will change when using other yield criterion. Any pressure dependence of yield (or dependence on the third stress invariant) will produce a different relationship between these two yield stresses. For this reason, you should insist on seeing the *actual lab data* from which the strength value was determined. This advice is true in general: always decide for yourself which theoretical model and properties are appropriate for a given application – don't trust anyone else to make this decision for you.

Side comment: Noting that J_2 is a function of stress, $\underset{\approx}{\boldsymbol{\sigma}}$, the geometry of a yield surface is often described by introducing a "yield function" as follows:

Yield function:	$f(\underset{\approx}{\boldsymbol{\sigma}}) := J_2 - k^2$
At yield (on the yield surface):	$f(\underset{\approx}{\boldsymbol{\sigma}}) = 0$
Below yield (in the elastic domain):	$f(\underset{\approx}{\boldsymbol{\sigma}}) < 0$

Yield functions are never unique. The yield function must only have the property that it is negative for all elastic states, zero for all states at yield, and positive for all states that cannot be reached through quasistatic elastic loading. For example, instead of $f(\underset{\approx}{\boldsymbol{\sigma}}) := J_2 - k^2$, we could have alternatively defined $f(\underset{\approx}{\boldsymbol{\sigma}}) := \sqrt{J_2} - k$, which satisfies the required sign conventions for a yield function. Another common choice for a von Mises yield function is $f(\underset{\approx}{\boldsymbol{\sigma}}) := \sigma_{\text{eq}} - Y$, where Y is the axial stress at yield in uniaxial stress loading, and σ_{eq} is an alternative stress invariant (called "equivalent stress" and almost always available for plotting in commercial FEM codes), defined by

$$\sigma_{\text{eq}} := \sqrt{\tfrac{3}{2}\underset{\approx}{\mathbf{s}} : \underset{\approx}{\mathbf{s}}} = \sqrt{3J_2}$$

A related strain invariant is

$$\varepsilon_{\text{eq}} := \sqrt{\tfrac{2}{3}\underset{\approx}{\boldsymbol{\gamma}} : \underset{\approx}{\boldsymbol{\gamma}}} \qquad \text{where} \qquad \underset{\approx}{\boldsymbol{\gamma}} = \text{dev}(\underset{\approx}{\boldsymbol{\varepsilon}})$$

Similarly, equivalent plastic strain is defined

$$\varepsilon_{\text{eq}}^{p} := \int_0^t \sqrt{\tfrac{2}{3}\underset{\approx}{\dot{\boldsymbol{\gamma}}}^{p} : \underset{\approx}{\dot{\boldsymbol{\gamma}}}^{p}}\, dt \qquad \text{where} \qquad \underset{\approx}{\dot{\boldsymbol{\gamma}}}^{p} = \text{dev}(\underset{\approx}{\dot{\boldsymbol{\varepsilon}}}^{p})$$

in which $\underset{\approx}{\dot{\boldsymbol{\varepsilon}}}^{p}$ is the plastic part of the strain rate, defined below.

Note that the equivalent stress is defined with a fraction 3/2, while equivalent strain is defined using the reciprocal of this fraction, 2/3. (Mnemonic: stress is big while strain is small, so stress gets the bigger fraction.) The fractions, 3/2 on stress but 2/3 on strain, are used so that σ_{eq} equals the axial component of stress under uniaxial stress loading and so that, for this and general loading, the plastic work "rate" is

$$\underset{\approx}{\boldsymbol{\sigma}} : \underset{\approx}{\dot{\boldsymbol{\varepsilon}}}^{p} = \sigma_{\text{eq}}\dot{\varepsilon}_{\text{eq}}^{p}$$

As is the case in any classical elastic-plastic model, the total strain rate, $\dot{\underset{\sim}{\underset{\sim}{\boldsymbol{\varepsilon}}}}$, is taken to be broken into two parts: an elastic (recoverable) part $\dot{\underset{\sim}{\underset{\sim}{\boldsymbol{\varepsilon}}}}^e$ plus a plastic (permanent/non-recoverable) part $\dot{\underset{\sim}{\underset{\sim}{\boldsymbol{\varepsilon}}}}^p$:

$$\dot{\underset{\sim}{\underset{\sim}{\boldsymbol{\varepsilon}}}} = \dot{\underset{\sim}{\underset{\sim}{\boldsymbol{\varepsilon}}}}^e + \dot{\underset{\sim}{\underset{\sim}{\boldsymbol{\varepsilon}}}}^p$$

The stress rate is taken to depend on the elastic strain rate according to elasticity theory. For simplicity, the upcoming algorithm presumes linear elasticity for which the bulk and shear modulus, K and G, are known material parameters. Accordingly,

$$\mathrm{dev}(\dot{\underset{\sim}{\underset{\sim}{\boldsymbol{\varepsilon}}}}^e) = \frac{1}{2G}\mathrm{dev}(\dot{\underset{\sim}{\underset{\sim}{\boldsymbol{\sigma}}}})$$
$$\mathrm{iso}(\dot{\underset{\sim}{\underset{\sim}{\boldsymbol{\varepsilon}}}}^e) = \frac{1}{3K}\mathrm{iso}(\dot{\underset{\sim}{\underset{\sim}{\boldsymbol{\sigma}}}})$$

During plastic loading, the stress is constrained to be on the yield surface (the region outside the yield surface is unattainable). Therefore, during plastic loading, the stress rate must either remain stationary at a single point on the yield surface, or it is allowed to move around *tangentially* to other states on the yield surface. An examination of full equations of plasticity (see the previously referenced documents on the CSM website) reveals that, *for strain-controlled loading*, the stress rate will be exactly what it would have been under Hooke's law except that the part of this "trial elastic stress rate" that is normal to the yield surface is discarded. This view leads to the following computational algorithm.

Below is a J_2 plasticity radial return algorithm …

A typical strain-controlled von Mises plasticity model is set up so that the inputs and outputs are

INPUTS:

$\dot{\underset{\approx}{\boldsymbol{\varepsilon}}}$ = total strain rate

Δt =time step

$\underset{\approx}{\boldsymbol{\sigma}}^{\text{beg}}$ = stress at the beginning of the time step

K = bulk modulus

G = shear modulus

τ_y = yield in shear $(= Y / \sqrt{3})$

OUTPUTS:

$\dot{\underset{\approx}{\boldsymbol{\varepsilon}}}^e$ = elastic strain rate

$\dot{\underset{\approx}{\boldsymbol{\varepsilon}}}^p$ = plastic strain rate

$\underset{\approx}{\boldsymbol{\sigma}}^{\text{end}}$ = stress at the end of the time step

The solution procedure is summarized as follows:

1. Tentatively assume elastic response to obtain a tentative "trial" prediction for the updated stress, found by $\underset{\approx}{\boldsymbol{\sigma}}^{\text{trial}} = \underset{\approx}{\boldsymbol{\sigma}}^{\text{beg}} + \Delta t\left[2G\ \text{dev}(\dot{\underset{\approx}{\boldsymbol{\varepsilon}}}) + 3K\,\text{iso}(\dot{\underset{\approx}{\boldsymbol{\varepsilon}}})\right]$. Compute its invariant J_2^{trial}.
2. Test if the trial stress is outside the yield surface, and complete the solution as follows:
   a. If $J_2^{\text{trial}} \le \tau_y^2$, then the tentative assumption of elasticity was correct because it predicted a stress not outside the yield surface. Accordingly, the updated stress is equal to the trial stress. Moreover, during elastic loading, the plastic strain rate is zero, and the elastic strain rate is therefore equal to the total strain rate.
   b. Otherwise, if $J_2^{\text{trial}} > \tau_y^2$, then the trial stress is outside the yield surface, implying that the tentative assumption of elasticity was incorrect. In this case, set the updated stress to be identical to the trial stress except with the magnitude of the stress deviator scaled down by a factor selected to put this modified stress exactly on the yield surface, thus giving the value for the corrected updated stress at the end of the step, $\underset{\approx}{\boldsymbol{\sigma}}^{\text{end}}$. The effective (consistent) stress rate for the step is then $\dot{\underset{\approx}{\boldsymbol{\sigma}}} \approx \dfrac{\underset{\approx}{\boldsymbol{\sigma}}^{\text{end}} - \underset{\approx}{\boldsymbol{\sigma}}^{\text{beg}}}{\Delta t}$. Applying Hooke's law then gives the elastic strain rate. Applying the strain rate decomposition then gives the plastic strain rate.

This solution scheme, in which the updated stress is found by simply scaling down the trial stress deviator, is called "radial return." *It gives grossly inaccurate results for the majority of more realistic (non von Mises) models.* Moreover, radial return applies in strain-control, but not stress-control. See http://csm.mech.utah.edu/content/wp-content/uploads/2011/09/2007BrannonPlasticityBookChapterWithErrata.pdf for the correct solution procedure for more sophisticated plasticity models. Any serious study of computational plasticity must be based on more complete and rigorously justified governing equations than what is provided here in this primer.

Pseudo code for the above algorithm is as follows

```
function J2plasticity(
     //inputs:
       edot, dt, sigBEG, K, G, tauy,
     //outputs:
       eEdot, ePdot,sigEND)

// ------------------------------------------------------------------
// Get the trial updated stress
// In what follows, "iso" and "dev" stand for functions that return
// the isotropic and deviatoric parts of a tensor, respectively

sigTRIAL = sigBEG + (2G*dev(edot) + 3K*iso(edot))*dt

// ------------------------------------------------------------------
// The stress state is above yield if J2>tauy^2.
// Equivalently, the stress state is above yield if ||dev(sigTRIAL)||
// is larger than sqrt(2)*tauy. Below, we test by looking at the ratio:

fac = ||dev(sigTRIAL)||/(sqrt(2)*tauy)

if(fac<=1)then
   //elastic

   sigEND=sigTRIAL
   ePdot = zeroTensor
   eEdot = edot

elseif (fac>1) then
   //plastic
   //set the updated stress to be the same as the trial stress
   //except that the stress deviator is reduced in magnitude to put the
   //updated stress on the yield surface (done by dividing by "fac" below)

   sigEND = iso(sigTRIAL)+dev(sigTRIAL)/fac

   //evaluate the consistent stress rate

   sigDOT = (sigEND-sigBEG)/dt
   //apply Hooke's law to get elastic strain rate:

   eEdot = iso(sigDOT)/(3K) + dev(sigDOT)/(2G)

   //apply strain rate decomposition to get the plastic strain rate
   ePdot = edot-eEdot

endif

// If desired, the updated equivalent stress and equivalent plastic strain rate are
sigEQUIV=sqrt(3/2)*||dev(sigEND)!!
ePdotEQUIV=sqrt(3/2)*||dev(ePdot)||  //update by ePequivEND = ePequivBEG*ePdotEQUIV*dt

// ------------------------------------------------------------------

end of function J2plasticity
```

Verification test #1: Uniaxial strain

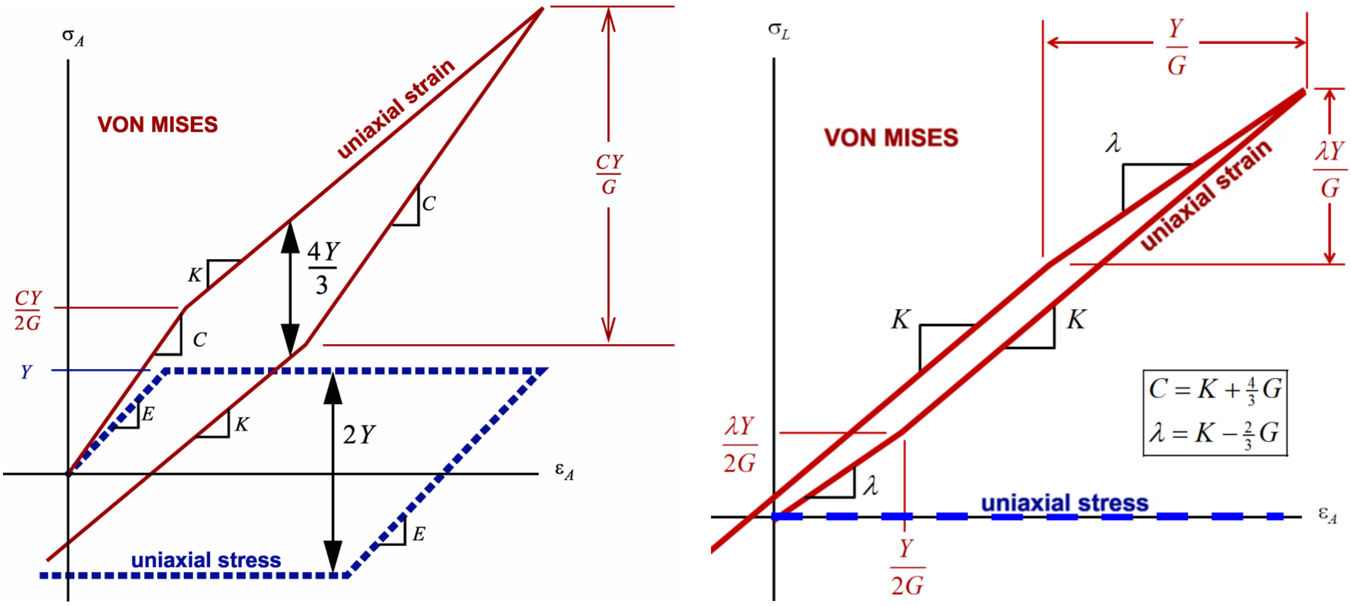
Under uniaxial strain, the axial component of strain, ε_A, increases with time while all other components of strain are zero. During the initial elastic phase of loading, Hooke's law implies that the axial stress σ_A and lateral stress σ_L are given by

$$\sigma_A = C\varepsilon_A, \quad \text{where} \quad C = K + \tfrac{4}{3}G = \text{"constrained modulus"}$$
$$\sigma_L = \lambda\varepsilon_A, \quad \text{where} \quad \lambda = K - \tfrac{2}{3}G = \text{"Lame}' \text{ modulus"}$$

Substituting these into the von Mises yield criterion shows that the yield criterion is reached when

$$\sigma_A^{\text{yield}} = \sigma^{\text{HEL}} = \frac{CY}{2G}, \quad \text{where} \quad Y = \sqrt{3}\,\tau_y$$

The label "HEL" stands for "Hugoniot Elastic Limit," where the word "Hugoniot" refers to states that can be reached in shock loading. (Shock waves in isotropic media are always initially uniaxial strain.) After yield is reached in uniaxial strain loading, the stress state moves along the surface of the von Mises cylinder in a direction parallel to the hydrostat, making the slope of the stress-strain plot equal to the bulk modulus, as indicated below. Unloading follows the elastic slope until the yield surface is once again reached, launching another phase with slope equal to the bulk modulus. The goal of this verification test is to reproduce the following plots of axial and lateral stresses, each shown as functions of the axial component of total strain.



Note from the figure that uniaxial strain is very different from uniaxial stress. In both cases, yielding causes the material to flow like a fluid. For uniaxial stress, the fluid is unconstrained laterally, causing zero change in axial stress to sustain flow. For uniaxial strain, the lateral constraint causes the slope of both axial and lateral components to be the bulk modulus during plastic flow. Both types of loading produce permanent plastic strains upon removal of the axial stress (and relieving the axial stress does not also relieve lateral stress in the uniaxial strain case).

WARNING: uniaxial stress is not a strain-controlled loading, so radial return does not apply to the tentative elastic solution. See the book chapter for details on the distinction between a test elastic stress and trial elastic stress!

Verification test #2: pure isochoric strain rates in different directions

This verification test initially applies a pure shear strain rate, and then it changes to a new strain rate that is still isochoric, but orthogonal to the initial strain rate. This causes the stress state to move around the von Mises cylinder without changing pressure (i.e., without moving parallel to the cylinder axis).

Table 2 Example 2: material parameters

Parameter	Value
Yield in shear, τ_y	165 MPa
Shear modulus, G	79 GPa

Example 2: strain table

Time (s)	ε_{11}	ε_{22}	ε_{33}
0	0	0	0
1	−0.003	−0.003	0.006
2	−0.0103923	0	0.0103923

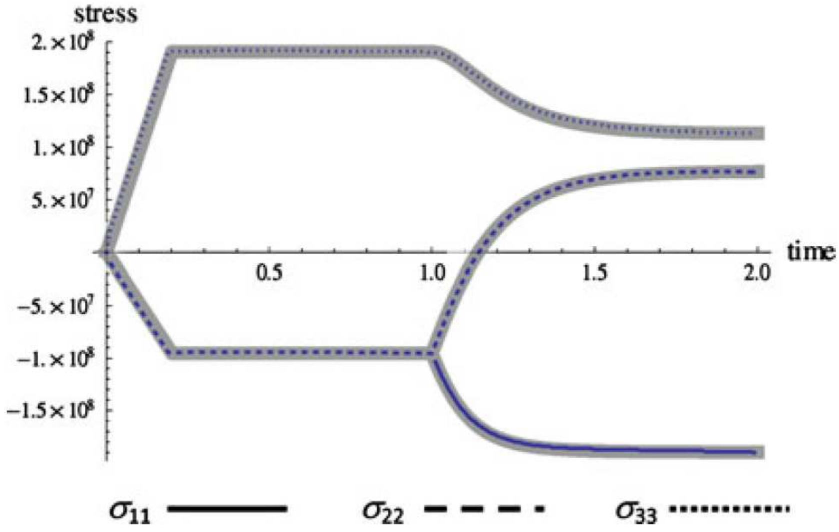


Fig. 13 The solution to the von Mises plasticity problem defined in Example 2. The *thick colored lines* are the analytical solution. The *Thin black lines* that overlay the exact solution a results from a a user-defined routine with nested return algorithm implemented in LS-DYNA

Note: since the problem is isochoric, a value for the bulk modulus is not needed. When testing your model, you should try putting the bulk modulus to a large value, say $K = 10G$, to call attention to any spurious predictions of pressure (which usually means you are accidentally introducing some volume changes in your kinematics driver). For details of this and other plasticity verification problems, see http://www.mech.utah.edu/~brannon/pubs/7-2009BrannonLeelavanichkul-IJF.pdf.