$\sin\omega t = \cos(\omega t - \frac{\pi}{2}) = 1\angle{-\pi/2}$

$-\sin\omega t = \cos(\omega t + \frac{\pi}{2}) = 1\angle{+\pi/2}$

$\sin(\omega t + \theta) = 1\angle{\theta - \pi/2}$

$-\sin(\omega t + \theta) = 1\angle{-\theta + \pi/2}$ (1)

TYPES OF FLUCTUATING FORCES:

Reciprocating – vary in magnitude only
Rotating – " " direction "
Fluctuating – vary in both mag & dir.

Periodic ⇒ repeating
Harmonic ⇒ sinusoidal

FOURIER: Periodic = sum of harmonics

"1^{st} harmonic" = harmonic w/ same freq as periodic fnt. (AKA fundamental harmonic)

"2^{nd} harmonic" ⇒ freq = twice 1^{st}

Representations for disp., vel. & acc

$\mathrm{Re}\{\ \}$

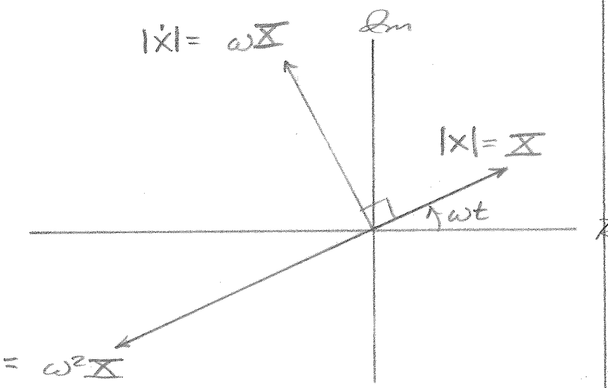
Disp $x = X\cos\omega t = Xe^{i\omega t} = X\angle{0}$

vel $\dot{x} = -\omega X\sin\omega t = \omega X\cos(\omega t + \pi/2) = i\omega X e^{i\omega t} = \omega X\angle{\pi/2}$

acc $\ddot{x} = -\omega^2 X\cos\omega t = \omega^2 X\cos(\omega t + \pi) = -\omega^2 X e^{i\omega t} = \omega^2 X\angle{\pi}$ (2)

where $A\angle{B} \equiv A\cos(\omega t + B)$ (3)

$B \rightarrow$ "phase angle"



PERIOD τ, FREQ f, ANG. FREQ ω

$\omega = \frac{2\pi}{\tau} = 2\pi f$ rad/s

$f = \frac{1}{\tau} = \omega/2\pi$ cycles/sec

$\tau = \frac{1}{f} = 2\pi/\omega$ sec (4)

PHASE

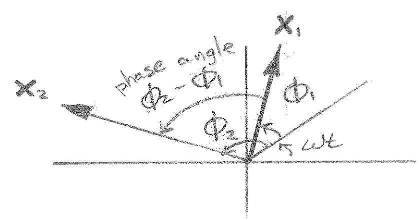
Given $x_1 = A_1\angle{\phi_1} \equiv A_1\cos(\omega t + \phi_1)$ (5)

$x_2 = A_2\angle{\phi_2}$

Phase angle $\equiv \phi_2 - \phi_1$ [=] rad (6)

Phase difference = *phase angle*/ω [=] sec

ADDITION OF HARMONICS



$x_1 = A_1\cos\omega t \qquad x_2 = A_2\sin\omega t = A_2\angle{-\pi/2}$ (7)

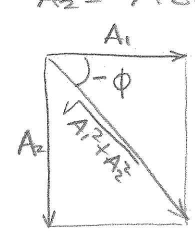
$A_1\cos\omega t + A_2\sin\omega t = A\cos(\omega t - \phi)$

or $A_1\angle{0} + A_2\angle{-\pi/2} = A\angle{-\phi}$ where:

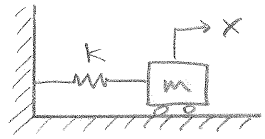
$\Big\}$ CBS $A_1 = A\cos\phi$, $A_2 = A\sin\phi$ (8)

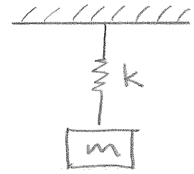
$A_1\cos\omega t + A_2\sin\omega t = A\cos(\omega t - \phi)$ where →	$A = \sqrt{A_1^2 + A_2^2}$ & $\phi = \tan^{-1}\frac{A_2}{A_1}$

(9)



1 DOF UNDAMPED SYSTEMS





EOM:

$$m\ddot{x} + kx = 0 \qquad (1)$$

Sol'n

$$x = A\cos(\omega t + \gamma) = B_1 \cos\omega t + B_2 \sin\omega t \qquad (2)$$

STATICAL DEFL: $\boxed{\delta_{st} = \frac{mg}{k}}$ (3)

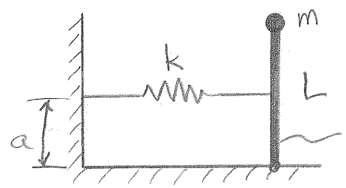
where $\boxed{\omega = \text{NATURAL FREQ} = \sqrt{\frac{k}{m}}}$ (4)

$\boxed{B_1 = A\cos\gamma \quad B_2 = A\sin\gamma \quad \text{or} \quad A = \sqrt{B_1^2 + B_2^2} \quad \gamma = \tan^{-1} B_2/B_1}$ (5)

W/ BC's, sol'n is

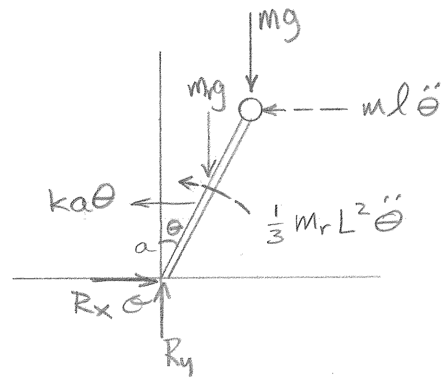
$$x = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t \qquad (6)$$

INVERTED PENDULUM



mass of rod = m_r
$\Rightarrow I = \frac{1}{3} m_r L^2$

Assume small displ. !!



(7)

Sum moments about O

$$-mg(l\theta) - m_r g(\tfrac{1}{2}l\theta) + ka\theta(a) + ml^2\ddot{\theta} + \tfrac{1}{3}m_r L^2\ddot{\theta} = 0 \qquad (8)$$

$$\Rightarrow \boxed{\begin{aligned} M_{ef} &= \tfrac{1}{3}m_r L^2 + ml^2 \\ k_{ef} &= ka^2 - (m + \tfrac{1}{2}m_r)gl \end{aligned}} \qquad (9)$$

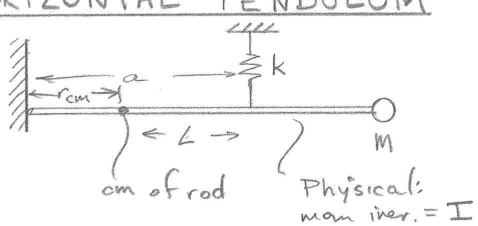
$\Rightarrow \omega_n^2 = \frac{k_{ef}}{m_{ef}}$ Must be > 0 ! (10)

$\Rightarrow$ require

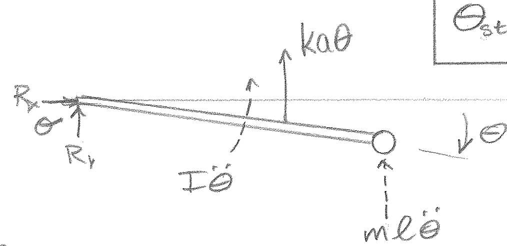
$$\boxed{k > (m + \tfrac{1}{2}m_r)gl/a^2} \qquad (11)$$

If spng weaker than this pendulum will continue to fall → no oscillation

HORIZONTAL PENDULUM

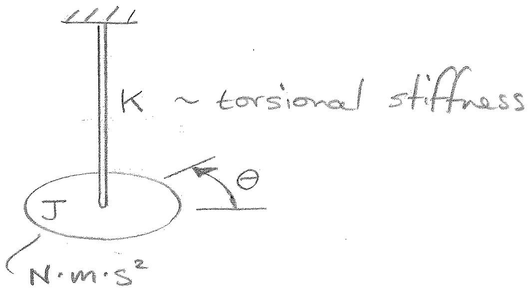


$\boxed{\theta_{st} = \frac{mgl + m_r g r_c}{ka^2}}$ (12)



$$\boxed{\begin{aligned} M_{ef} &= ml^2 + I \\ K_{ef} &= ka^2 \end{aligned}} \Rightarrow \omega_n^2 = \frac{k_{ef}}{M_{ef}} \qquad (13)$$

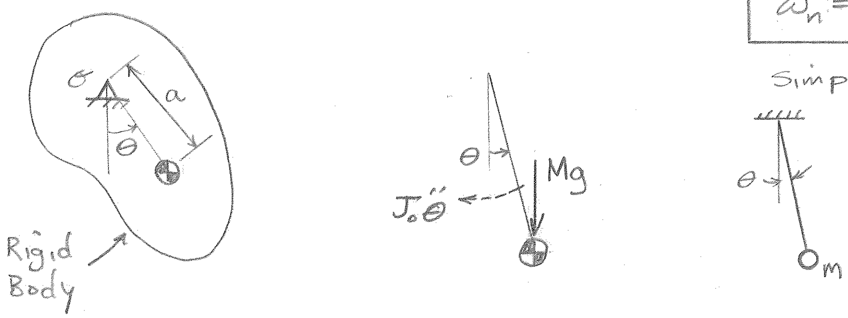
TORSIONAL PENDULUM



$$J\ddot{\theta} + k\theta = 0 \tag{1}$$

COMPOUND (PHYSICAL) PENDULUM

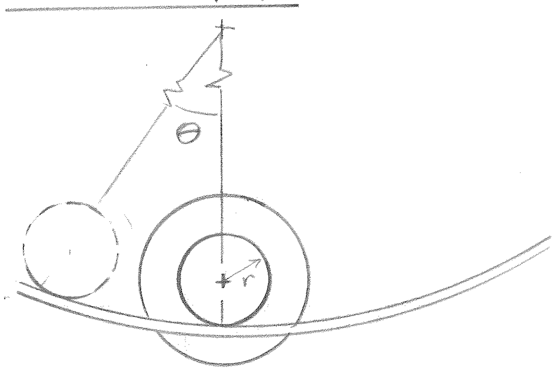
$$\boxed{\begin{aligned} J_o\ddot{\theta} + Mga\theta &= 0 \\ \omega_n^2 &= \frac{Mga}{J} \end{aligned}} \tag{2}$$



Simple pendulum: $M=m$ $J=ml^2$, $a=l$ (3)

$$\boxed{\omega^2 = \frac{g}{l}} \tag{4}$$

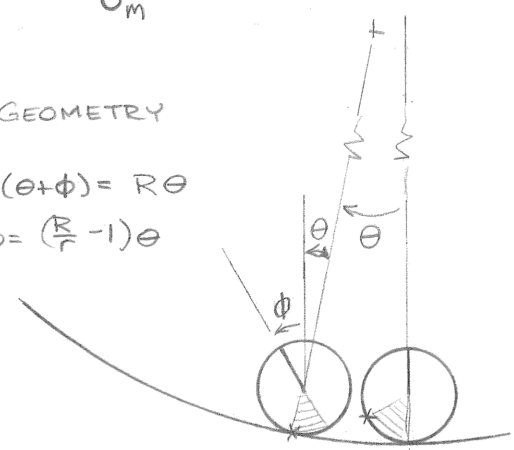
DISK ON RAIL



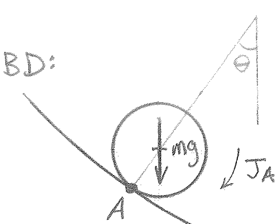
GEOMETRY

$$r(\theta+\phi) = R\theta$$

$$\phi = \left(\frac{R}{r}-1\right)\theta \tag{5}$$



FBD:



$$J_A = J_o + Md^2 = \tfrac{1}{2}mr^2 + mr^2 = \tfrac{3}{2}mr^2 \tag{6}$$

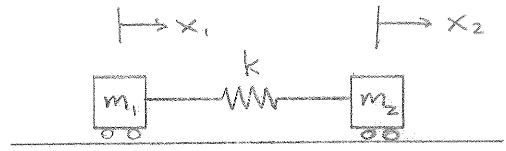
$$mg(r\underbrace{\sin\theta}_{\approx\theta}) + J_A\ddot{\Phi} = 0 \tag{7}$$

$$\hookrightarrow \left(\frac{R}{r}-1\right)\ddot{\theta}$$

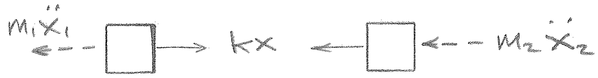
$$\left(\frac{R}{r}-1\right)J_A\ddot{\Phi} + mgr\theta = 0 \tag{8}$$

$$\Rightarrow \omega_n^2 = \frac{2g}{3(R-r)} \tag{9}$$

DEGENERATE SYSTEM



FBD'S



Denote $x = x_2 - x_1$ (1)

$$m_1\ddot{x}_1 - kx = 0 \qquad\Rightarrow\qquad m_1 m_2 \ddot{x}_1 - k m_2 x = 0$$
$$m_2\ddot{x}_2 + kx = 0 \qquad\qquad\quad\; m_1 m_2 \ddot{x}_2 + k m_1 x = 0$$
$$m_1 m_2 \underbrace{(\ddot{x}_2 - \ddot{x}_1)}_{\ddot{x}} + k(m_1 + m_2)x = 0 \qquad (2)$$

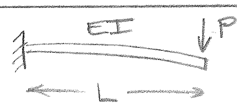
So $\ddot{x} + \omega_n^2 x = 0$ (3)

$$\boxed{\omega_n^2 = \frac{k}{m_1} + \frac{k}{m_2}} \qquad (4)$$

REMARK: There is also a rigid body mode ($x=0$) for which $\omega_n = 0$

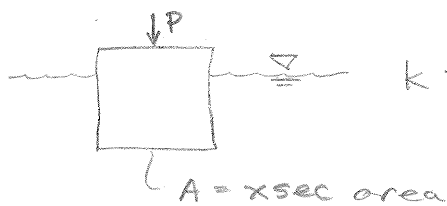
EQUIVALENT STIFFNESSES

$k = F/\delta$ or M/θ (5)

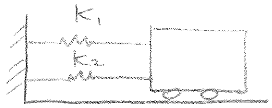


$$k = \frac{3EI}{L^3} \qquad (6)$$

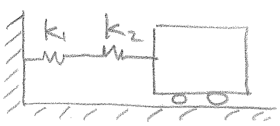
$$k = \frac{GJ}{L} \qquad J = \tfrac{1}{2}\pi r^4 = \pi d^4/32 \qquad (7)$$



$k = \gamma A$ $\quad \gamma$ = specific weight of water (8)

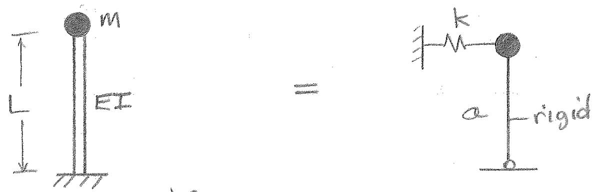


$$k = k_1 + k_2 \qquad (9)$$

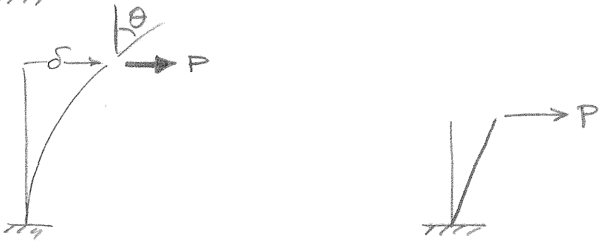


$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \qquad (10)$$

EQUIVALENT SYSTEMS



Find k & a for the two systems to be dynamically equivalent with respect to the translation and rotation of the mass m.



$\delta = PL^3/3EI$ $\qquad \delta = P/k$ $\qquad \Rightarrow k = 3EI/L^3$ (1)

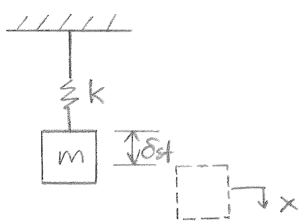
$\theta = PL^2/2EI$ $\qquad \theta = \delta/a$ $\qquad \Rightarrow a = \dfrac{\delta}{\theta} = \dfrac{P/k}{PL^2/2EI}$ (2)

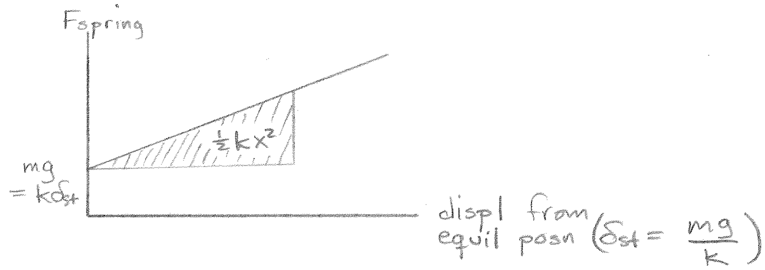
$= \frac{2}{3}L$

ENERGY METHOD FOR NATURAL FREQ

$T + U = \text{const}$ ⟶ gives governing DE (3)

$T_{max} = U_{max}$ ⟶ gives natural freq (4)





(5)

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \text{const} \qquad (6)$$

Differentiate & cancel $\dot{x}$'s.

$$m\ddot{x} + kx = 0 \qquad (7)$$

To directly get ω_n^2, write $T_{max} = U_{max}$ (8)

$$T_{max} = \tfrac{1}{2}m\dot{x}_{max}^2 = \tfrac{1}{2}k\,x_{max}^2 = U_{max} \qquad (9)$$

see fig. p 1.

$$\omega_n = \frac{\dot{x}_{max}}{x_{max}} = \sqrt{\frac{k}{m}} \qquad (10)$$

Proof: $\left(\dfrac{\dot{x}_{max}}{x_{max}}\right)^2 = \left(\dfrac{\|\dot{X}\|}{\|X\|}\right)^2 = \omega^2$

EXAMPLE NEXT PAGE

EFFECTIVE MASS

When the motion of every pnt in a multimass or distrib. mass is known, then you can find M_{ef} ⇒

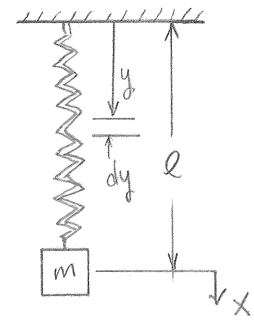
$$T = \tfrac{1}{2}M_{eff}\,\dot{x}^2 \qquad (11)$$

so $\qquad \omega_n^2 = K_{eff}/M_{eff}$ (12)

To find M_{ef}, you must know (or assume) the amplitude distribution.

EXAMPLE: ACCOUNTING FOR SPRING MASS

Assume velocity distribution is linear



$$v = \dot{x}\,\frac{y}{\ell} \qquad (1)$$

Then k.e. is

$$T = \tfrac{1}{2} m \dot{x}^2 + \int_0^{\ell} \tfrac{1}{2}\left(\dot{x}\,\tfrac{y}{\ell}\right)^2 \underbrace{dm}_{\rho dy}$$

$$= \tfrac{1}{2} m \dot{x}^2 + \frac{\rho \dot{x}^2}{2\ell^2}\int_0^{\ell} y^2\,dy$$

$$= \tfrac{1}{2} m \dot{x}^2 + \frac{\rho \dot{x}}{6}\,\ell$$

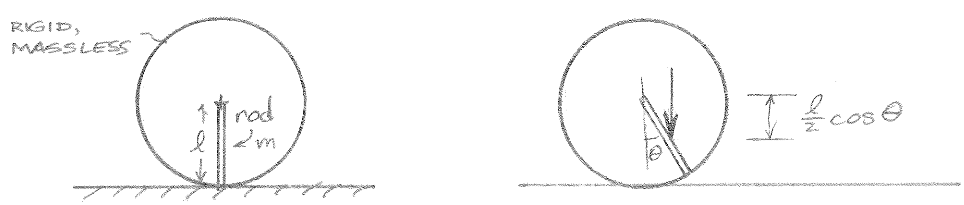
$$= \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{6} m_s \dot{x}^2 \;\overset{\text{set}}{=}\; \tfrac{1}{2} m_{eff} \dot{x}^2 \qquad (2)$$

$$\Rightarrow \quad m_{eff} = m + \tfrac{1}{3} m_s \qquad (3)$$

So the revised frequency is

$$\omega^2 = \frac{k}{m + \frac{1}{3} m_s} \qquad (4)$$

EXAMPLE: ENERGY METHOD



This problem is perfect for energy methods because the only position for which the rod's moment of inertia about the IC is known exactly is when the rod is vertical down, which is the posn of max K.E.

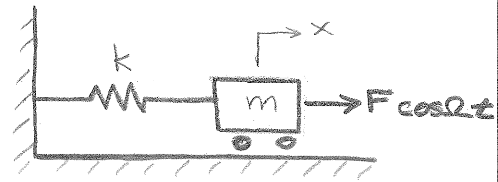
$$E = T_{max} = \tfrac{1}{2} I_{IC}\,\dot{\theta}^2 = \tfrac{1}{2}\left(\tfrac{1}{3} m \ell^2\right)\dot{\theta}^2_{max} \qquad (5)$$

$$E = V_{max} = mg\,\tfrac{\ell}{2}\underbrace{(1-\cos\theta_{max})}_{\approx \frac{\theta^2_{max}}{2}} \qquad (6)$$

$$\omega_n^2 = \frac{\dot{\theta}^2_{max}}{\theta^2_{max}} = \frac{3g}{2\ell} \qquad \Rightarrow \qquad \ddot{\theta} + \frac{3g}{2\ell}\,\theta = 0 \qquad (7)$$

FORCED HARMONIC VIBRATION

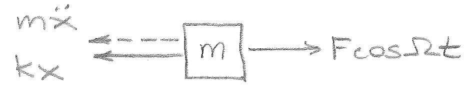
$$m\ddot{x} + kx = F\cos\Omega t$$ (1)



$$x = x_H + x_p$$ (2)

$$x_H = A\cos(\omega_n t - \gamma) = B_1\cos\omega_n t + B_2\sin\omega_n t$$ (3)

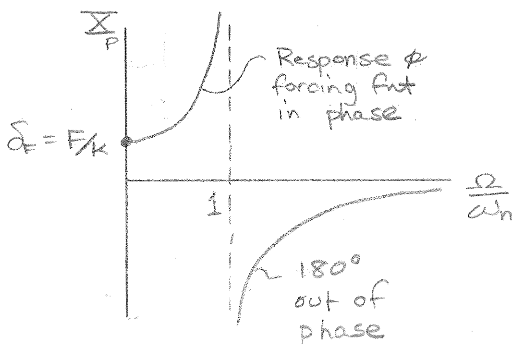
$$\begin{cases} x_p = X_p\cos\Omega t \quad (+\cancel{Y\sin\Omega t}) \\ \ddot{x}_p = -\Omega^2 X_p\cos\Omega t \end{cases}$$ (4)

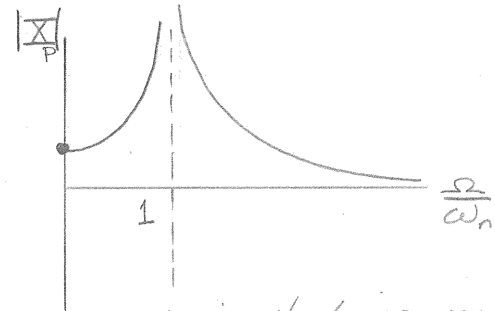


$$\Rightarrow \quad (k - m\Omega^2)X_p = F$$ (5)

$\Omega \neq \omega_n$ necessarily

$$\boxed{X_p = \frac{F}{k - m\Omega^2}}$$ (6)





This is what we usually plot.

TOTAL SOL'N

$$x = B_1\cos\omega_n t + B_2\sin\omega_n t + \underbrace{\frac{F}{k - m\Omega^2}}_{X_p}\cos\Omega t$$ (7)

IC's

$$x(0) = x_0 = B_1 + X_p \quad \Rightarrow \quad B_1 = x_0 - X_p$$
$$\dot{x}(0) = \dot{x}_0 = B_2\,\omega_n \qquad\quad B_2 = \dot{x}_0/\omega_n$$

$$\frac{F/m}{\omega_n^2 - \Omega^2} = \frac{F/k}{1 - (\Omega/\omega_n)^2}$$ (8)

Then

FORCED EOM ($\Omega \neq \omega_n$)

$$\boxed{x = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{F/m}{\omega_n^2 - \Omega^2}(\cos\Omega t - \cos\omega_n t)}$$ (9)

For resonant case ($\Omega = \omega_n$), write RHS_3 as:

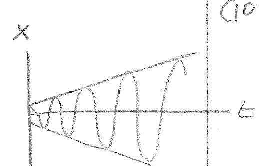
$$RHS_3 = \frac{F}{m}\lim_{\Omega\to\omega_n}\frac{\cos\Omega t - \cos\omega_n t}{\omega_n^2 - \Omega^2} = \frac{F}{m}\lim_{\Omega\to\omega_n}\frac{-t\sin\Omega t}{-2\Omega} = \frac{F}{2m}\frac{t\sin\omega_n t}{\omega_n} = \frac{F}{2\sqrt{km}}t\sin\omega_n t$$

Then

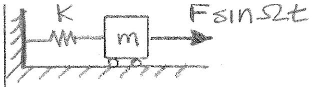
RESONANCE EOM ($\Omega = \omega_n$)

$$\boxed{x = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{F}{2\sqrt{km}}\,t\sin\omega_n t}$$ (10)

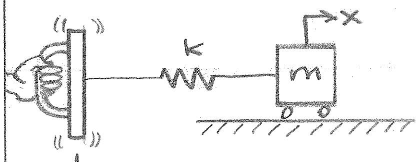
grows w/o bound!



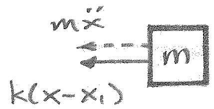
useful rel'ns: $\dfrac{\omega_n}{k} = \dfrac{1}{\sqrt{km}} = \dfrac{1}{m\omega_n}$

	$x = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t$	$+ \quad \frac{F/\sqrt{km}}{\omega_n^2-\Omega^2}\left[\omega_n\sin\Omega t - \Omega\sin\omega_n t\right]$	8

HARMONIC DISPLACEMENT EXCITATION



$x_1 = D\cos\Omega t$

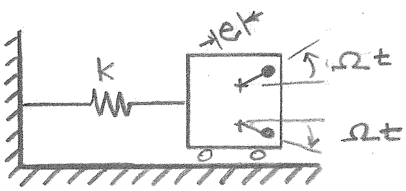


$$m\ddot{x} + k(x-x_1) = 0 \quad \Rightarrow \quad m\ddot{x} + kx = \underbrace{kD}_{F^*}\cos\omega t \qquad (1)$$

Ah! Just replace F in the force sol'ns by kD.

$$\boxed{F :\to kD} \qquad (2)$$

HARMONIC ROTATING IMBALANCE EXCITATION

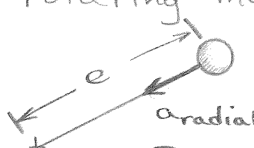


m = total mass = car mass + 2 rotating masses

W = weight of each rotating mass,

m_r = total mass of the imbalances $= 2W/g$

FBD: rotating mass



$a_{radial} = e\Omega^2$

$\Rightarrow F_{radial} = \frac{W}{g}e\Omega^2$

$\Rightarrow$ Force on block due to two rotating imbalances is

$$\boxed{f(t) = m_r e\Omega^2\cos\Omega t} \qquad (3)$$

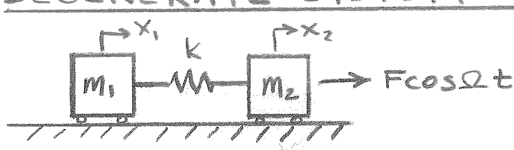
And pos'n of imbalances is

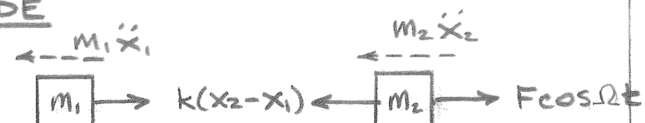
$$x_r = x + e\cos\omega t \qquad (4)$$

Ah!

$$\boxed{F :\to m_r e\Omega^2} = \underbrace{(\overset{\text{total!}}{W_r} e)}_{\text{often just this product is given.}}\frac{\Omega^2}{g}$$

DEGENERATE SYSTEM - RB MODE





Let $x \equiv x_2 - x_1$

FORCE BALANCE:

$$\begin{array}{rl} (-m_1\ddot{x}_1 + kx = 0) & m_2 \\ + \quad (m_2\ddot{x}_2 + kx = F\cos\Omega t) & m_1 \\ \hline m_1 m_2(\ddot{x}_2 - \ddot{x}_1) + k(m_1+m_2) = m_1 F\cos\Omega t & \end{array} \qquad (5)$$

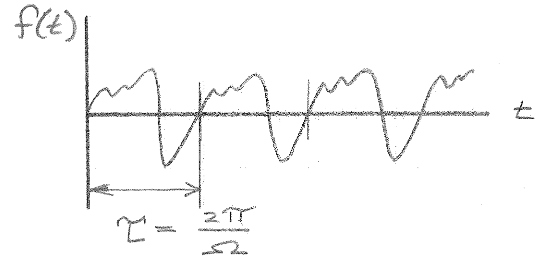
Rearrange:

$$kx + \underbrace{\left[\frac{1}{m_1} + \frac{1}{m_2}\right]^{-1}}_{m_{eq}} x = \underbrace{\left[\frac{F}{1+m_2/m_1}\right]}_{F_{eq}}\cos\Omega t \qquad (6)$$

So $\omega_n^2 = k/m_{eq}$ (7)

And $\boxed{\frac{F}{m} :\to \frac{F_{eq}}{m_{eq}} == \frac{F}{m_2}}$ (8)

PERIODIC FORCE EXCITATION



Write a Fourier series

$$f(t) = a_1 \sin\Omega t + a_2 \sin 2\Omega t + \cdots$$
$$+ b_0 + b_1 \cos\Omega t + b_2 \cos 2\Omega t + \cdots \qquad (1)$$

Or, for a force,

$$\boxed{F = F_0 + F_1\cos(\Omega t - \gamma_1) + F_2\cos(2\Omega t - \gamma_2) + \cdots}$$

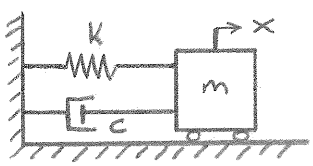
RESPONSE $x = X_0 + X_1\cos(\Omega t - \gamma_1) + X_2\cos(2\Omega t - \gamma_2) + \cdots$

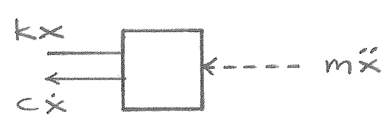
where

$$X_n = \frac{F_n}{k - m(n\Omega)^2} \qquad (2)$$

$$\Rightarrow \quad x = X_0 + \sum_{n=1}^{\infty} \frac{F_n}{k - m(n\Omega)^2}\cos(n\Omega t - \gamma_n)$$

VISCOUS DAMPING $\Rightarrow$ damping force $\propto$ velocity





DE: $\boxed{m\ddot{x} + c\dot{x} + kx = 0}$

Try a sol'n of the form $x = e^{st}$. Then (3)

$$ms^2 + cs + k = 0 \qquad (4)$$

$$\Rightarrow \quad s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad \Rightarrow \quad x = C_1 e^{s_1 t} + C_2 e^{s_2 t} \qquad (5)$$
$$= -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$$
$$= -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - c_c^2}$$
$$= \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

CRITICAL DAMPING FACTOR

$$\boxed{c_c \equiv 2\sqrt{mk} = 2m\omega_n = 2k/\omega_n} \qquad (6)$$

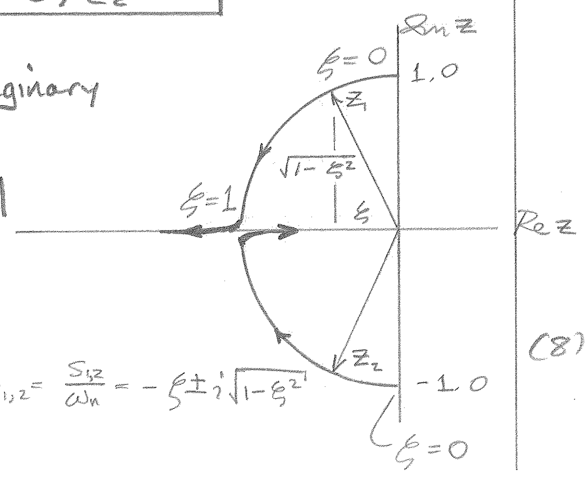
DAMPING RATIO

$$\boxed{\zeta \equiv c/c_c} \qquad (7)$$

OSCILLATORY $\zeta < 1$, underdamped, $s_{1,2}$ = imaginary

CRITICAL $\zeta = 1$, $s_1 = s_2 = -\omega_n$

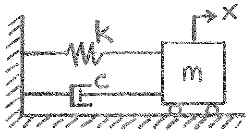
NONOSCIL. $\zeta > 1$, overdamped, $s_{1,2}$ = real

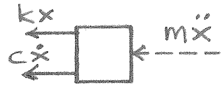


Useful relns: $\frac{c}{k} = 2\zeta/\omega_n$ (8)

$$z_{1,2} = \frac{s_{1,2}}{\omega_n} = -\zeta \pm i\sqrt{1-\zeta^2}$$

VISCOUS DAMPING





GOVERNING DE: $\boxed{m\ddot{x} + c\dot{x} + kx = 0}$

SOL'N: $x = C_1 e^{s_1 t} + C_2 e^{s_2 t}$

$s_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$

$\omega_n = \sqrt{k/m}$

"damping ratio": $\zeta = c/c_c$

"critical damping factor": $c_c = 2\sqrt{mk} = 2m\omega_n = 2k/\omega_n$

OSCILLATORY $\zeta < 1$ UNDERDAMPED, PERIODIC

$s_{1,2} = \omega_n\left(-\zeta \pm i\sqrt{1-\zeta^2}\right)$ ← complex!

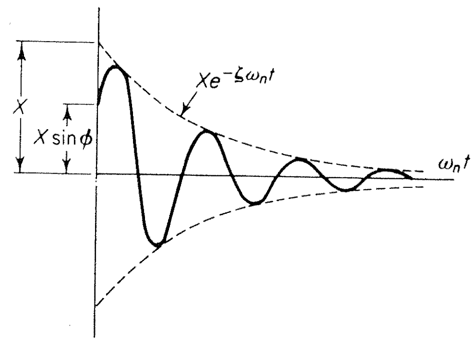
FREQUENCY OF DAMPED OSCILATION: $\boxed{\omega_d \equiv \omega_n\sqrt{1-\zeta^2}}$

SOL'N: $x = e^{-\zeta\omega_n t}\left(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right)$

$= X e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$

$= e^{-\zeta\omega_n t}\left(B_1 \sin\omega_d t + B_2 \cos\omega_d t\right)$

$B_1 = \dfrac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}$ $B_2 = x_0$



CRITICALLY DAMPED $\zeta = 1$

$s_1 = s_2 = -\omega_n$ ← double root!

SOLN: $x = e^{-\omega_n t}\left(B_1 t + B_0\right)$

$B_1 = \dot{x}_0 + \omega_n x_0$ $B_0 = x_0$

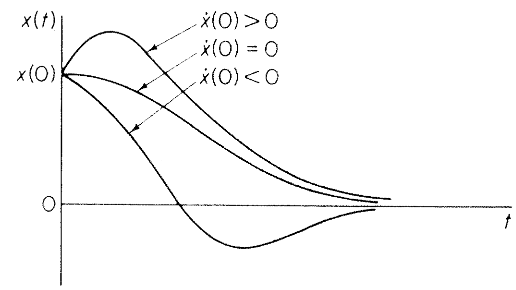


Figure 2.3-5. Critically damped motion $\zeta = 1.0$.

NONOSCILLATORY $\zeta > 1$ OVERDAMPED, APERIODIC

$s_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2-1}\right)$ ← real!

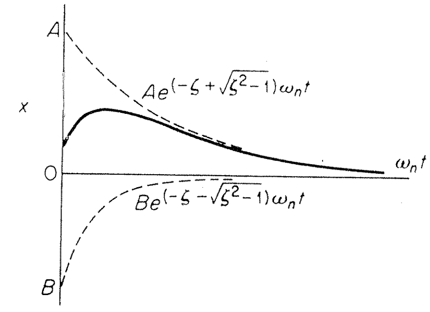
SOL'N: $x = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$

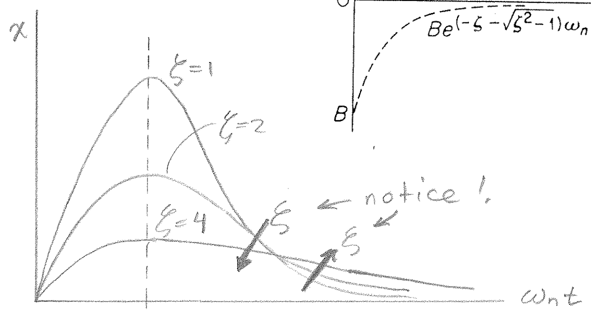
$C_1 = \dfrac{\dot{x}_0 + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$

$C_2 = \dfrac{-\dot{x}_0 - \left(\zeta - \sqrt{\zeta^2-1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$

For $x_0 = 0$,

$x = \dfrac{\dot{x}_0 e^{-\zeta\omega_n t}}{\omega_n\sqrt{\zeta^2-1}} \sinh\left(\sqrt{\zeta^2-1}\,\omega_n t\right)$





LOGARITHMIC DECREMENT

A way to measure the amt of damping ζ in a system is to measure the rate of decay of free oscillations.

Logarithmic decrement $\delta = \ln$ (ratio of any two sucessive amplitudes)

$$\boxed{\delta = \ln \frac{x_0}{x_1}} = \ln \frac{x(t_1)}{x(t_1+\tau_d)} = \ln \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+\tau_d)}} = \zeta\omega_n\tau_d \qquad (1)$$

Recall

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad (2)$$

Then

$$\boxed{\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta \quad \text{for small } \zeta} \qquad (3)$$

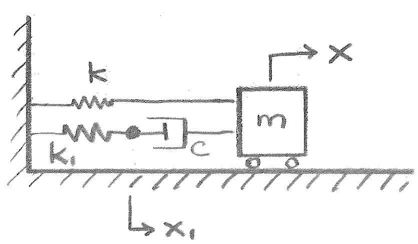
If δ is measured xpmly, then the damping ratio can be found by

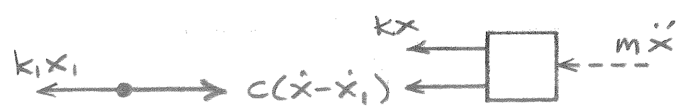
$$\boxed{\zeta = \sqrt{\frac{1}{1+(2\pi/\delta)^2}} \approx \frac{\delta}{2\pi} \quad \text{for small } \zeta} \qquad (4)$$

The logarithmic decrement can also be given in terms of the amplitude ratio after n cycles:

$$\boxed{\delta = \frac{1}{n} \ln \frac{x_0}{x_n}} \qquad \textit{Proof:} \quad \frac{x_0}{x_n} = \frac{x_0}{x_1}\frac{x_1}{x_2}\cdots\frac{x_{n-1}}{x_n} = (e^{\delta})^n \qquad (5)$$

1½ DOF SYSTEM





We get a system:

① $k_1 x_1 = c(\dot{x}-\dot{x}_1)$

② $m\ddot{x} + c(\dot{x}-x_1) + kx = 0$ (6)

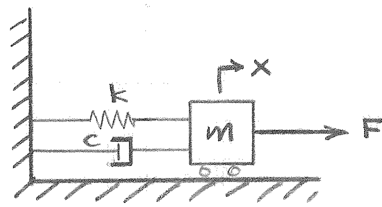
Put ① in ② then $\frac{d}{dt}$ & solve for $\dot{x}_1$, put result in ②...

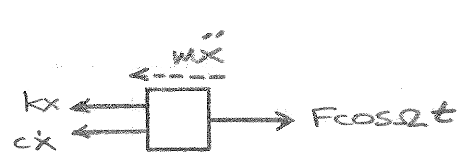
$$m\ddot{x} + c\dot{x} + \frac{c}{k_1}(m\dddot{x} + k\dot{x}) + k\dot{x} = 0 \qquad (7)$$

This system is called 1½ DOF because it is governed by a 3rd order ODE (1DOF ⇒ 2nd Ⓞ ODE, 2DOF ⇒ 4th Ⓞ ODE)

FORCED VIBRATION WITH DAMPING

NOTE "ss" ⇒ steady state because $x_H \to 0$ as $t \to \infty$

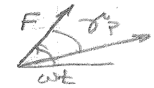




$$m\ddot{x}+c\dot{x}+kx=F\cos\Omega t \quad \text{or} \quad \ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=\frac{F}{m}\cos\Omega t \qquad (1)$$

The homogeneous soln is given by 9-5. Assume a particular soln of the form

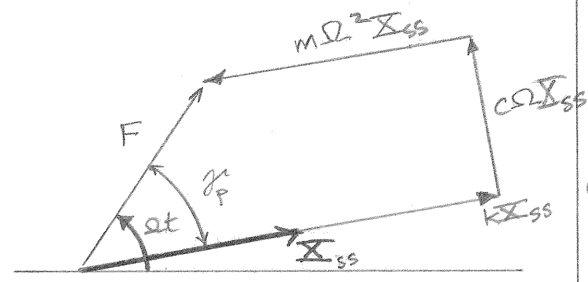
$$x_{ss}=X_{ss}\cos(\Omega t-\gamma_{ss}) \qquad (2)$$



Substitute (2) in (1). Recalling (p.1) the vector relnships b/w disp., vel, & acc., this eqn for X_p & γ_p can be represented graphically:
Then

$$X_{ss}=\frac{F}{\sqrt{(k-m\Omega^2)^2+(c\Omega)^2}} \qquad (3)$$

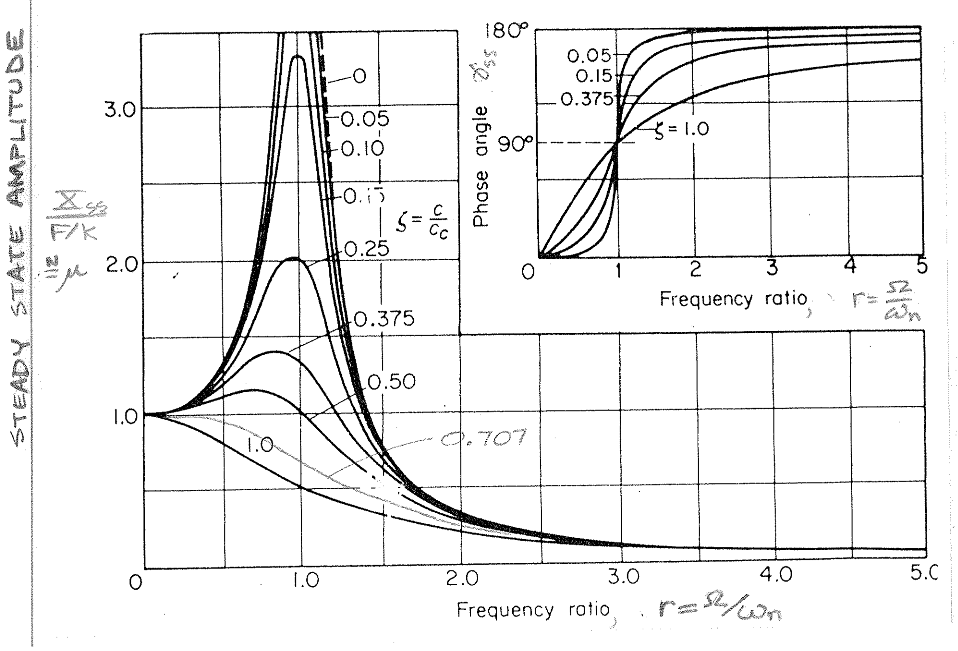
$$\gamma_{ss}=\tan^{-1}\frac{c\Omega}{k-m\Omega^2} \qquad (4)$$

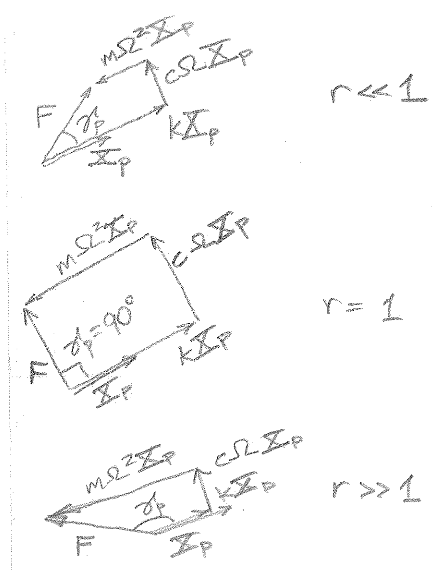


These can be put in the nondimensional form

$\dfrac{X_{ss}}{F/k}=\left\{(1-r^2)^2+(2\zeta r)^2\right\}^{-1/2}$	$\omega_n=\sqrt{k/m}$
	$C_c=2\sqrt{mk}=2m\omega_n$
	$\zeta=c/C_c$
$\tan\gamma_{ss}=\dfrac{2\zeta r}{1-r^2}$	$r=\Omega/\omega_n$

(5)





TOTAL SOL'N: FORCED VIBRATION W/ DAMPING

Combine the homogeneous soln (9-5) and the particular soln (12-2 & 12-5) $\Rightarrow x = x_H + x_P$. If $\xi < 1$ the homogeneous soln is harmonic and may be written

$$x_H = A e^{-\xi\omega_n t}\cos(\omega_n t - \gamma') \qquad (1)$$

So that the total soln is

$$\boxed{m\ddot{x} + c\dot{x} + kx = F\cos\Omega t}$$

$$\boxed{x = A e^{-\xi\omega_n t}\cos(\omega_d t - \gamma') + X_{ss}\cos(\Omega t - \gamma_{ss})} \qquad (2)$$

where $X_{ss} = \dfrac{F/k}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$

$\dfrac{F}{k} = \dfrac{F}{m\omega_n^2}$

$\gamma_{ss} = \tan^{-1}\dfrac{2\xi r}{1-r^2}$

$\omega_n = \sqrt{k/m}$

$\omega_d = \sqrt{1-\xi^2}\,\omega_n$

$c_c = 2\sqrt{mk} = 2m\omega_n$

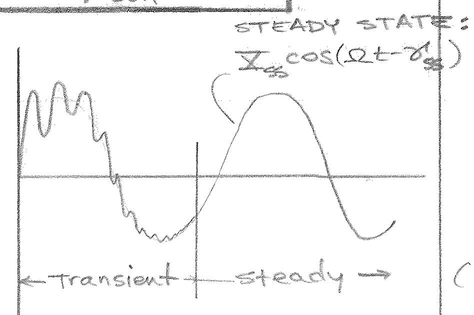
$\xi = c/c_c$

$r = \Omega/\omega_n$

$\dfrac{c\Omega}{k} = 2\xi r$ (3)

The qualitative behavior of this soln is as sketched. The homogeneous contribution to 13-2 superposes a transient response onto the steady state soln given by

$$x_{ss} = X_{ss}\cos(\Omega t - \gamma_{ss}) \qquad (4)$$



HOW does X_{ss} vary w/ ξ & $r\ (= \Omega/\omega_n)$? Local resonance:

Maximizing eqn 12-5 implies that X_{ss} is max when

$$r\left[r^2 - (1-2\xi^2)\right] = 0 \qquad (5)$$

If $r=0$, then $\Omega = 0$ so that we have <u>no</u> oscillations, trivial.

$$\Rightarrow \quad r = \sqrt{1-2\xi^2} \qquad (6)$$

since we require $r\ (= \Omega/\omega_n)$ to be real, we see that X_{ss} has a <u>local</u> max only if $1 - 2\xi^2 > 0$, or

$$\xi < \frac{1}{\sqrt{2}} = 0.707 \qquad (7)$$

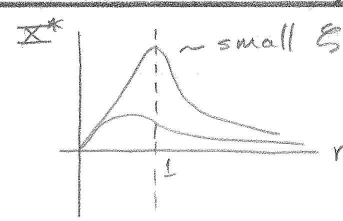
The max local amplitude is found by using (6) in (3a):

$$\text{GIVEN } \xi, \quad X_{ss})_{max} = \frac{F/k}{2\xi\sqrt{1-\xi^2}} \quad \text{at } r^2 = 1 - 2\xi^2 \qquad (8)$$

EXPERIMENTAL DETERMINATION OF DAMPING FACTOR, C

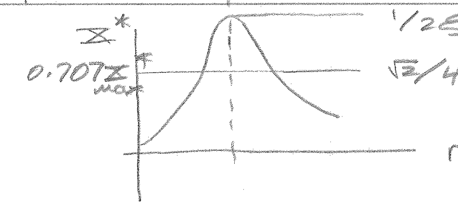
For small damping, (8) becomes

$$X^*_{max} = \frac{x_{ss})_{max}}{F/k} \approx \frac{1}{2\xi} \qquad (9)$$



For small ξ, X^* is approx. symmetric about $r = 1$. Define "bandwidth" by $r_2 - r_1$ where

r_2 & r_1 correspond to an amplitude 0.707 the peak.
From (13-3a), we seek solns to



$$\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}} = \frac{\sqrt{2}}{4\xi} \qquad (1)$$

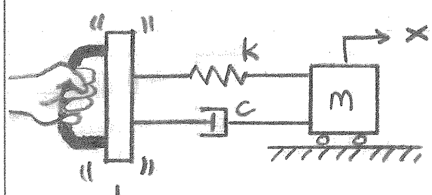
$$\Rightarrow\Rightarrow \quad r_{1,2}^2 = 1 - 2\xi^2 \mp 2\xi\sqrt{1+\xi^2} \qquad (2)$$

$$\Rightarrow \quad r_2^2 - r_1^2 = 4\xi\sqrt{1+\xi^2} \approx 4\xi = (r_2-r_1)(r_2+r_1) \qquad (3)$$

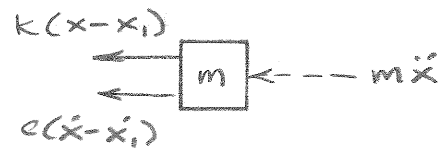
For near symmetry about $r=1$, $r_2+r_1 = [(1+\Delta r)+(1-\Delta r)] = 2$ (4)
Hence $4\xi = 2(r_2-r_1)$, or (5)

$$\boxed{\xi = \tfrac{1}{2}(r_2-r_1)} \qquad (6)$$

FORCED DISPL. EXCITATION W/ DAMPING



$\hookrightarrow x_1 = D\cos\Omega t$



$$m\ddot{x} + c(\dot{x}-\dot{x}_1) + k(x-x_1) = 0 \qquad (7)$$

Let $x_2 = x - x_1$. Then (8)

$$m\ddot{x}_2 + c x_2 + k x_2 = -m\ddot{x}_1 = +m\Omega^2 D\cos\Omega t \qquad (9)$$

Steady state soln (from 13-2 w/ $F = m\Omega^2 D$)

$$x_2 = X_2\cos(\omega t - \gamma_2) \qquad (10)$$

$$X_2 = \frac{m\Omega^2 D/k}{\sqrt{(1-r^2)^2+(2\xi r)^2}} = \frac{r^2 D}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \qquad (11)$$

$$\gamma_2 = \tan^{-1}\frac{2\xi r}{1-r^2} \qquad (12)$$

The soln for x, rather than x_2, is $x = x_1 + x_2$, or

$$\boxed{x = D\cos\Omega t + X_2\cos(\omega t - \gamma_2)} \qquad (13)$$

Alternative Sol'n: don't introduce x_2; write (7) as

$$m\ddot{x} + c\dot{x} + kx = c\dot{x}_1 + kx_1 = -c\Omega D\sin\Omega t + kD\cos\Omega t \qquad (14)$$
$$= \hat{X}\cos(\Omega t - \hat{\gamma})$$

where (see eq 1-9) $\hat{X} = D\sqrt{k^2 + c^2\Omega^2}$, $\hat{\gamma} = \tan^{-1}\left(-\frac{c\Omega}{k}\right) = \tan^{-1}(-2\xi r)$ (15)

With the change of variable,

$$\hat{t} = t - \frac{\hat{\gamma}}{\Omega}, \tag{1}$$

the ODE becomes

$$m x'' + c x' + k x = \hat{X} \cos \Omega \hat{t} \qquad x' \equiv \frac{dx}{d\hat{t}}, \quad x'' \equiv \frac{d^2x}{d\hat{t}^2} \tag{2}$$

which is of the same form as 13-2. Hence, the particular sol'n is

$$x_p = X \cos(\Omega \hat{t} - \gamma_p) = \boxed{X \cos(\Omega t - \beta)} \qquad \beta \equiv \hat{\gamma} + \gamma_p \tag{3}$$

where

$$X = \frac{\hat{X}/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \overset{(14\text{-}15)}{=} D \underbrace{\left\{ \frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right\}^{1/2}} \tag{4}$$

$$\gamma_p = \tan^{-1} \frac{2\zeta r}{1-r^2}$$

we call this "TRANSMISSIBILITY" $\mathcal{K}$

$$\boxed{\mathcal{K} \equiv \frac{X}{D}} \tag{5}$$

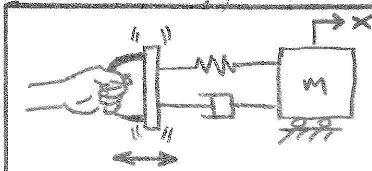
$$\beta = \hat{\gamma} + \gamma_p$$

$$= \tan^{-1}(-2\zeta r) + \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$

$$\overset{?}{=} \tan^{-1} \left[\frac{-2\zeta r + \frac{2\zeta r}{1-r^2}}{1 + \frac{4\zeta^2 r^2}{1-r^2}} \right]$$

$$= \tan^{-1} \left[\frac{2\zeta r^3}{1 - r^2 + 4\zeta^2 r^2} \right] \qquad \text{gawd!} \tag{6}$$

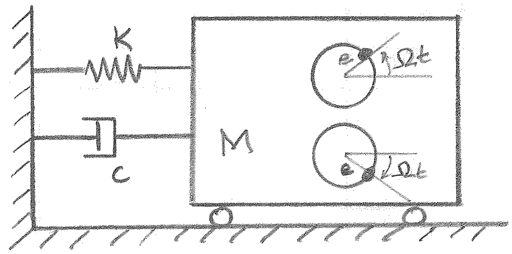
In summary,

 $D\cos\Omega t$ DISPL. EXCITATION !	Response: $x = \mathcal{K} D \cos(\Omega t - \beta)$ where $\mathcal{K} = \sqrt{\dfrac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}$ $\beta = \tan^{-1}\left(\dfrac{2\zeta r^3}{1-r^2+4\zeta^2 r^2}\right)$

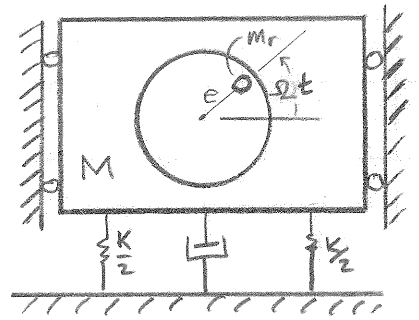
(7)

$\mathcal{K}$ = "transmissibility"

ROTATING UNBALANCE



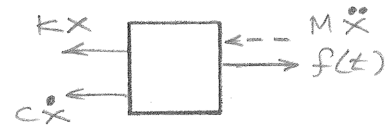
or



e = eccentricity

m_r = rotating mass (total)

M = Total mass = $m_{cart} + m_r$



Recall (p8) this pbm was solved for an undamped system, and we found that the harmonic force due to the eccentricity was

$$f(t) = m_r e \Omega^2 \cos \Omega t \tag{1}$$

So force balance is

$$M\ddot{x} + c\dot{x} + kx = m_r e \Omega^2 \cos \Omega t \tag{2}$$

From 13-2, $\quad x_{ss} = X_{ss} \cos(\Omega t - \gamma_{ss}) \qquad (3)$

$$X_{ss} = \frac{\frac{m_r e}{M}\, r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad \gamma_{ss} = \tan^{-1} \frac{2\zeta r}{1-r^2} \tag{4}$$

(where $r = \Omega/\omega_n$)

For a rotating imbalance, it is common for $\Omega \gg \omega_n$, so that r is very large:

$$X_\infty \overset{N}{=} \lim_{r \to \infty} X_p = \frac{m_r e}{M} \tag{5}$$

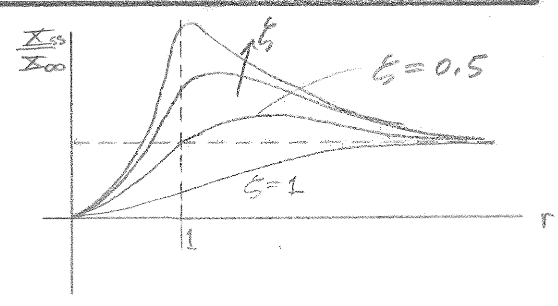
NOTE $\gamma_{ss} = \pi/2$ when $r = 1$

So the ss soln is

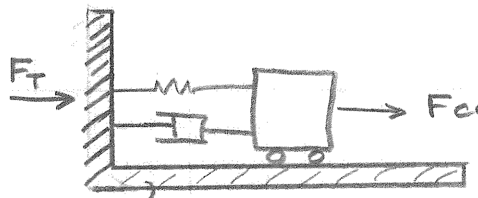
$x_{ss} = X_{ss} \cos(\Omega t - \gamma_{ss})$ $X_{ss} = r^2 \mu X_\infty$ $\gamma_{ss} = \tan^{-1} \frac{2\zeta r}{1-r^2}$	where μ is the standard amplification factor $\mu = \{(1-r^2)^2 + (2\zeta r)^2\}^{-1/2}$ and $X_\infty = m_r e / M \qquad r = \frac{\Omega}{\omega_n}$

(6)

see Text p 54 for a graph of X_{ss}/X_∞ & γ_{ss}. The X_{ss}/X_∞ curves now peak out to the right of r=1 and the start at zero and end at unity. The γ_{ss} curves are unchanged.



VIBRATION ISOLATION

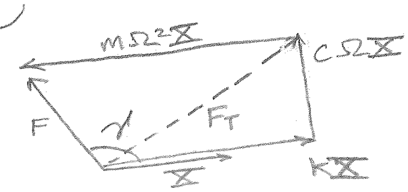


TRANSMITTED FORCE:

$$F_T = \sqrt{(kX)^2 + (c\Omega X)^2}$$ (1)

$$= kX\sqrt{1 + (2\zeta r)^2}$$ (2)

Eq 13-3a



"TRANSMISSIBILITY" $\mathcal{K}$

$$\boxed{\mathcal{K} \equiv \frac{F_T}{F} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}} \quad \left(r = \frac{\Omega}{\omega_n}\right)$$ (3)

NOTICE: Same as eq 15-6 !

See Thomson p 64 for plots.

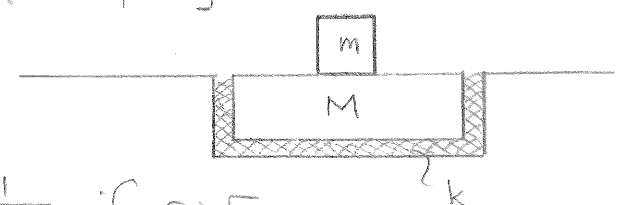
$$\mathcal{K} = \frac{\text{support force}}{\text{applied force}} = \frac{\text{response disp}}{\text{support disp}}$$ (4)

OBSERVATIONS

1) $\mathcal{K} < 1$ only for $r > \sqrt{2}$ ⇒ isolation possible only for $r > \sqrt{2}$
2) When $r > \sqrt{2}$, an undamped spring isolates better than a damped spring. ($c \downarrow \zeta \uparrow$)

ISOLATION PADS

Idea: support machine m on Block M. For ζ = small,



$$\mathcal{K} \simeq \frac{1}{\sqrt{(1-r^2)^2}} = \frac{1}{|1-r^2|} = \frac{1}{r^2-1} \quad \text{if } r > \sqrt{2}$$ (5)

The necessary frequency to obtain a certain transmissibility can be found by

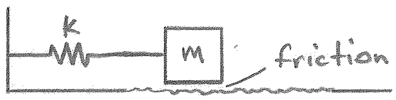
$$f = 15.76\sqrt{\frac{1}{\Delta}\left(\frac{2-R}{1-R}\right)} \quad \text{Hz}$$ (6)

Δ = statical defl.
$= g/\omega_n^2 = \sqrt{\frac{k}{M+m}}$

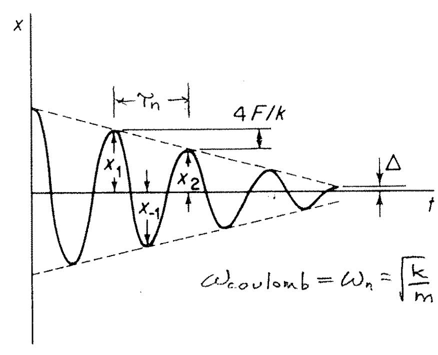
R = "reduction in transmissibility"

$R \equiv 1 - \mathcal{K}$ (7)

COULOMB DAMPING



Constant force always opposing motion.
Work = ΔK.E. Start at max extension thru one half cycle



$$-F(X_i + X_{i+\frac{1}{2}}) = \frac{1}{2}k(X_{i+\frac{1}{2}}^2 - X_i^2)$$ (8)

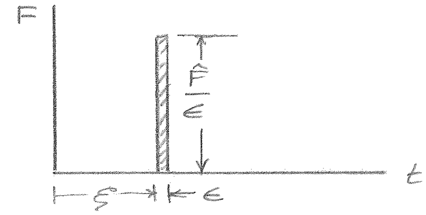
$$\Rightarrow X_{i+1/2} - X_i = -\frac{2F}{k}$$ (9)

$\omega_{coulomb} = \omega_n = \sqrt{\frac{k}{m}}$

$$\Rightarrow \boxed{\text{Decrease in amplitude per } \underline{\text{full}} \text{ cycle} = 4F/k}$$ (10)

CBS: frequency = ω_n (11)

TRANSIENT VIBRATION



IMPULSE

Impulse $\hat{F} = \int F(t)\,dt$ (1)

$\hat{F}$ finite, $F(t)$ large, dt small

DELTA FNT $\delta(t-\xi)$:

$$\int_0^\infty f(t)\,\delta(t-\xi)\,dt = f(\xi)\,H[t-\xi] \qquad (2)$$

Impulse momentum thm $F\,dt = m\,dv \Rightarrow$ step change in velocity:

$$\Delta v = \hat{F}/m \qquad (3)$$

w/o a change in disp.

FREE VIBRATION $x = x_0 \cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t$ (4)

$x_0 = 0, \quad \dot{x}_0 = \hat{F}/m$ (5)

$$\boxed{x = \frac{\hat{F}}{m\omega_n}\sin\omega_n t} \qquad (6)$$

VISCOUSLY DAMPED $x = X e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi\right)$ (7)

$x_0 = 0 \quad \Rightarrow \quad \phi = 0$ (8)

$$\dot{x}_0 = \frac{\hat{F}}{m} = X e^{-\zeta\omega_n t}\left[\sqrt{1-\zeta^2}\,\omega_n \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) - \zeta\omega_n \sin\sqrt{1-\zeta^2}\,\omega_n t\right]_{t=0}$$

$$= X\omega_n\sqrt{1-\zeta^2}$$

viscous $\zeta < 1$

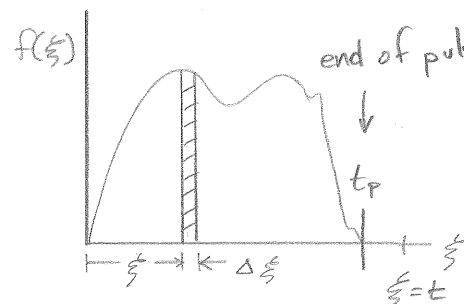
$$\Rightarrow \quad \boxed{x = \frac{\hat{F}}{m\omega_n\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)} \qquad (9)$$

"UNIT IMPULSE RESPONSE" $g(t)$

$$g(t) \equiv \frac{x(t)}{\hat{F}} \quad \Rightarrow \quad x = \hat{F}\,g(t) \qquad (10)$$

← get $g(t)$ from (6) or (9)

ARBITRARY EXCITATION



Consider arbitrary force to be a series of impulses.

$$\hat{F} = f(\xi)\,\Delta\xi \qquad (11)$$

Response depends on time elapsed since application of $\hat{F}$. From (10),

$$\text{response} = \hat{F}\,g(t-\xi) = f(\xi)\,\Delta\xi\,g(t-\xi)$$

Since system is linear, superposition holds. Hence

CONVOLUTION INTEGRAL!

$$\boxed{x(t) = \int_0^t f(\xi)\,g(t-\xi)\,d\xi} \quad \text{or} \quad \boxed{x(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau} \qquad (12)$$

Using $\mathcal{G}(\xi)$ from (6) & from (9),

UNDAMPED

$$\boxed{x(t) = \int_0^t f(\xi)\,\frac{1}{m\omega_n}\sin\omega_n(t-\xi)\,d\xi = \int_0^t f(t-\tau)\,\frac{1}{m\omega_n}\sin\omega_n\tau\,d\tau} \quad (1)$$

VISCOUS

$$\boxed{\begin{aligned} x(t) &= \int_0^t f(\xi)\,\frac{e^{-\zeta\omega_n(t-\xi)}}{m\omega_n\sqrt{1-\zeta^2}}\sin\left[\sqrt{1-\zeta^2}\,\omega_n(t-\xi)\right]d\xi \\ &= \int_0^t f(t-\tau)\,\frac{e^{-\zeta\omega_n\tau}}{m\omega_n\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n\tau\right)d\tau \end{aligned}} \quad (2)$$

BASE EXCITATION

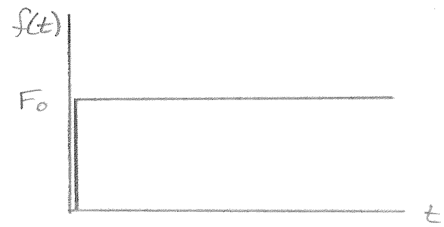
EOM: $\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\ddot{x}_B$ (3)

$\Rightarrow$ just replace f/m by $-\ddot{x}_B$. $\begin{cases} z = \text{relative disp} = x - x_B \\ \ddot{x}_B = \text{acceleration of base} \end{cases}$ (4)

So that

$$z(t) = \int_0^t -\ddot{x}_B(t-\tau)\,\frac{e^{-\zeta\omega_n\tau}}{\omega_n\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n\tau\right)d\tau \quad (5)$$

RESPONSE TO UNIT STEP FNT



$f(\xi) = F_0\,H[\xi]$ (6)

$\Rightarrow\ x(t) = \int_0^t F_0\,H[\xi]\,\mathcal{G}(t-\xi)\,d\xi$ (7)

$x(t) = F_0\int_0^t \mathcal{G}(t-\xi)\,d\xi$ (8)

UNDAMPED: $\mathcal{G}(\tau) = \frac{1}{m\omega_n}\sin\omega_n\tau$

$$\boxed{x = \frac{F_0}{k}(1-\cos\omega_n t)} \quad (9)$$

PEAK RESPONSE

$$\boxed{x_{peak} = 2F_0/k \quad \text{at} \quad t_{peak} = \pi/\omega_n} \quad (10)$$

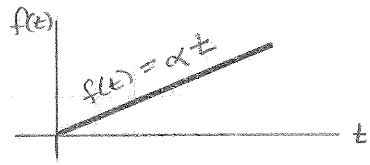
DAMPED: $\mathcal{G}(\tau) = \frac{e^{-\zeta\omega_n\tau}}{m\omega_n\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n\tau\right)$

See Thomson for derivation

$$\boxed{\begin{aligned} x &= \frac{F_0}{k}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \psi\right)\right] \\ \psi &= \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}} \end{aligned}} \quad (11)$$

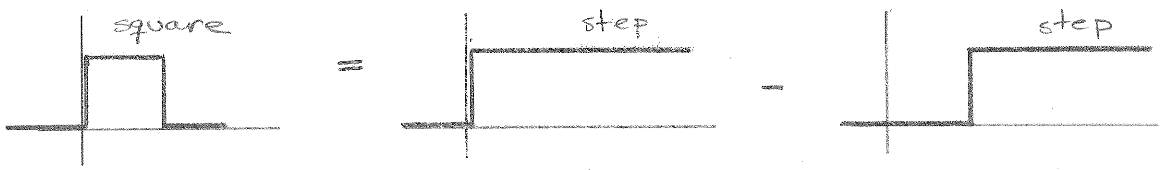
$$\boxed{t_{peak} = \frac{1}{\omega_n}\frac{\pi}{\sqrt{1-\zeta^2}}} \quad (12)$$

RESPONSE TO A RAMP (Undamped)

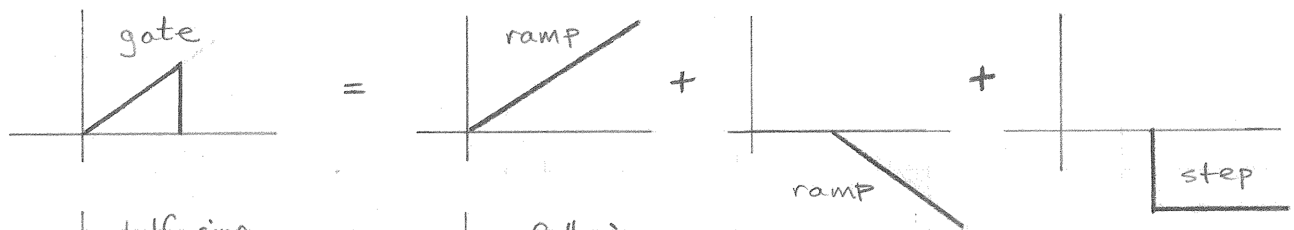


$$\boxed{x = \frac{\alpha}{k}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right)}$$ (1)

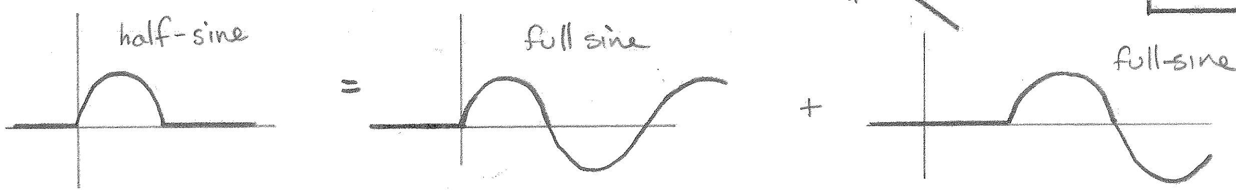
SUPERPOSITION



(2)

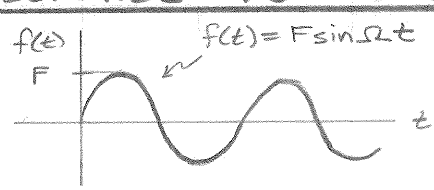


(3)



(4)

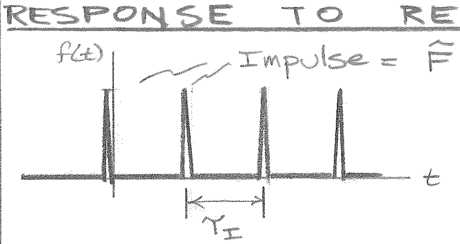
RESPONSE TO FULL SINE (Undamped)



$$\boxed{x = \frac{F/\sqrt{km}}{\omega_n^2 - \Omega^2}\left(\omega_n \sin\Omega t - \Omega\sin\omega_n t\right)}$$ (5)

Remark: this response can be derived using the methods of p. 7 + 8 or by the convolution integral

RESPONSE TO REPEATED IMPULSES (Undamped)



$$A = \frac{\hat{F}}{2\omega_n m \sin(\pi\omega_n/\Omega_I)}$$ (6)

$$\phi \overset{?}{=} \frac{\pi}{2} - \frac{\pi\omega_n}{\Omega_I}$$ (7)

Max spring force, F_s:

$$\frac{F_s\,\tau_I}{\hat{F}} = \frac{\pi\omega_n/\Omega_I}{\sin(\pi\omega_n/\Omega_I)}$$ (8)

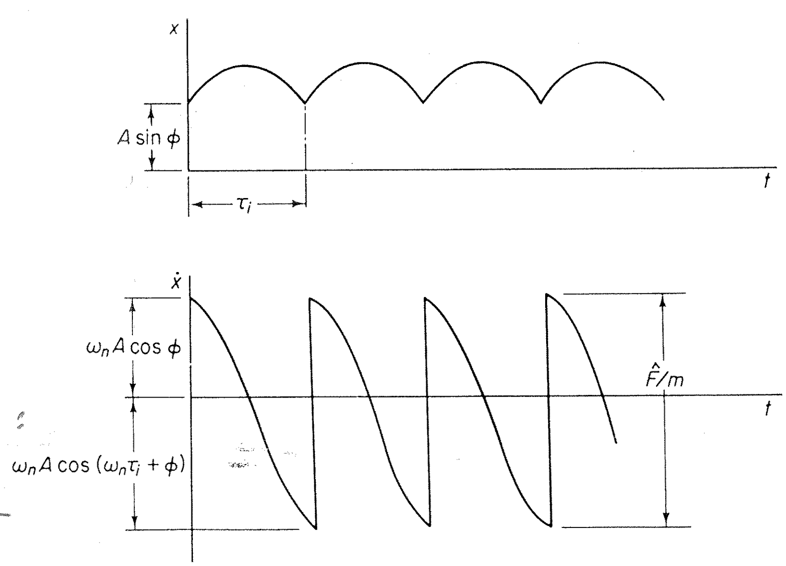


Figure 4.2-6. Displacement and velocity.

RESPONSE SPECTRUM

Shock = sudden application of a force

Max value of response characterizes severity of shock.

"RESPONSE SPECTRUM" = peak response v.s. ntrl freq

Response (18-12) $\quad x(t) = \int_0^t f(\xi)\,\mathcal{G}(t-\xi)\,d\xi \qquad \mathcal{G}(\tau) \equiv \dfrac{x(\tau)}{\hat{F}}$ (1)

where for a SDOF oscillator (18-6),

$$\mathcal{G}(\tau) = \frac{1}{m\omega_n}\sin\omega_n\tau \tag{2}$$

and for a damped oscillator (18-9),

$$\mathcal{G}(\tau) = \frac{e^{-\zeta\omega_n\tau}}{m\omega_n\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n\tau\right) \tag{3}$$

ASIDE: For support motion $f(\xi) :\to -m\underbrace{\ddot{y}(\xi)}_{\text{acc. of support.}}$

Peak response

$$x_{max} = \left|\int_0^t f(\xi)\,\mathcal{G}(t-\xi)\,d\xi\right|_{max} \tag{4}$$

"Dynamic Factor" (usually < 2)

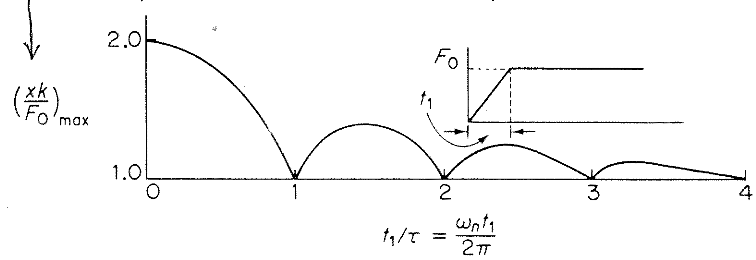


Figure 4.4-1.

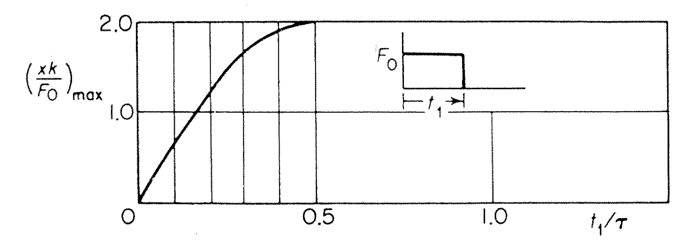
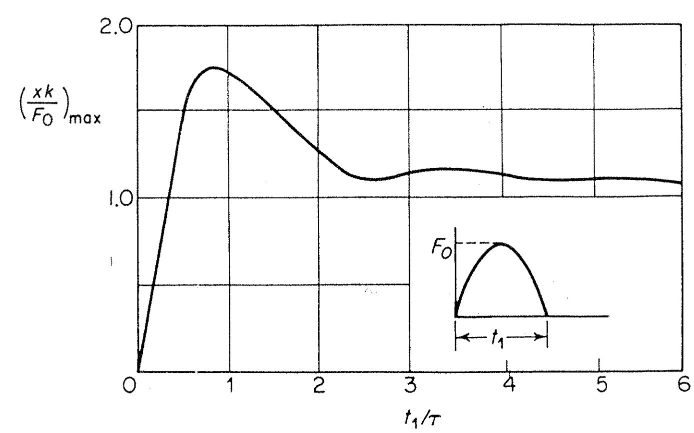


Figure 4.4-2.



PSEUDO RESPONSE SPECTRA : $\dot{x}_{max}$ v.s. ω_n

(Used in ground shock situations)

Obtain x_{max} & $\ddot{x}_{max}$ by $\quad x_{max} = \dot{x}_{max}/\omega_n$

$\ddot{x}_{max} = \dot{x}_{max}\,\omega_n$ (1)

"pseudo" because these are exact only if peak response occurs after shock pulse has passed, in which case the motion is harmonic

WORK & POWER

Consider $\quad f = F\cos\Omega t \quad$ & $\quad x = X\cos(\Omega t - \gamma)$ (2)

Work per cycle by external force — implies damping. This is SS soln (see 13-3)

$$W_{ext} = \int_{x_1}^{x_2} f\,dx = \int_{t_1}^{t_2} f\frac{dx}{dt}\,dt$$

$$= \int_0^{2\pi/\Omega} F\cos\Omega t\left[-X\Omega\sin(\Omega t-\gamma)\right]dt$$

$$= -FX\int_0^{2\pi}\cos(\Omega t)\sin[(\Omega t)-\gamma]\,d(\Omega t)$$

$$\boxed{W_{ext} = \pi F X\sin\gamma} \quad \text{per cycle} \qquad (3)$$

So $\left.\begin{array}{l}\gamma = 0 \text{ or } \pi \\ r \neq 1\end{array}\right\}$ (harmonic, little damping – see 13-3) $\quad W_{ext} = 0$

$\gamma = \pi/2 \Rightarrow \underbrace{r=1}_{\text{resonant freq.}} \quad W_{ext} = \pi F X \leftarrow$ max

Work by damping

$$W_d = \int -c\dot{x}\,dx = \int -c\dot{x}\underbrace{\left(\frac{dx}{dt}\right)}_{\dot{x}}dt = \int -c\dot{x}^2\,dt = \int_0^{2\pi} -c\Omega X^2\sin^2(\Omega t-\gamma)\,d(\Omega t)$$

$$\boxed{W_{damping} = -\pi c\Omega X^2} \quad \text{per cycle} \qquad (4)$$

Power consumption

phase angle

$$\text{Power} = \frac{\text{work}}{\text{time}} = \frac{\text{work}}{\text{cycle}}\Big/\underbrace{\frac{\text{time}}{\text{cycle}}}_{\frac{2\pi}{\Omega}} \overset{(3)}{=} \frac{\pi F X\sin\gamma}{2\pi/\Omega} = \frac{\Omega F X}{2}\sin\gamma \qquad (5)$$

$$\overset{13\text{-}3b}{=} \frac{\Omega F X}{2}\,\frac{2\zeta r}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \overset{13\text{-}3a}{=} \frac{\Omega F^2}{2k}\,\frac{2\zeta r}{(1-r^2)^2+(2\zeta r)^2} \qquad (6)$$

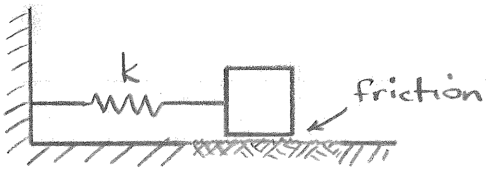
since $\Omega = r\omega_n = r\sqrt{k/m}$,

$$\boxed{\text{Power} = pf\,\frac{F^2}{\sqrt{km}}} \quad \boxed{\text{where } pf = \frac{\zeta r}{(1-r^2)^2+(2\zeta r)^2}} \quad (pf)_{max} = \frac{1}{4\zeta} \text{ when } r=1 \qquad (7)$$

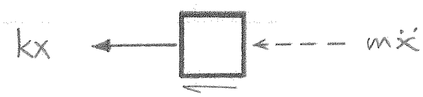
↳ max at $r=1$ $\qquad$ ↳ "Power factor"

Base excitation $F: \to -m\ddot{y} = m\Omega^2 Y$ (8)

COULOMB DAMPING REVISITED (see p17)



Damping force is constant



$f\,\underbrace{\mathrm{sgn}\,\dot{x}}_{-1+2H[\dot{x}]}$ (1) (2)

$$m\ddot{x} + kx + f\,\mathrm{sgn}\,\dot{x} = 0 \quad (3)$$

let $f = ka$ $\quad a$ = coef. of coulomb damping (4)

$$m\ddot{x} + k(x + a\,\mathrm{sgn}\,\dot{x}) = 0 \quad (5)$$

let $x^* = x + a\,\mathrm{sgn}\,\dot{x}$ (6)

$$\ddot{x}^* = \ddot{x}\,(1 + 2\delta[\dot{x}])$$

$$\left(\frac{m}{1+2\delta[\dot{x}]}\right)\ddot{x}^* + k\,x^* = 0 \quad (7)$$

Sol'n for $\dot{x} \neq 0$:

$$x^* = A\cos(\omega_n t - \gamma) \quad (8)$$

(6) ⇒ $\boxed{x = -a\,\mathrm{sgn}\,\dot{x} + A\cos(\omega_n t - \gamma)}$ sol'n for $\dot{x} \neq 0$ (9)

When $\dot{x} = 0$, require continuity of x & $\dot{x}$. (10)

Take $t=0$ to be the first point when $\dot{x} = 0$, $x(0) = \Delta_0$ ↗ $\gamma = 0$
Then $\mathrm{sgn}\,\dot{x} = -\mathrm{sgn}\,\Delta_0$, and

$\boxed{x(t) = a\,\mathrm{sgn}\,\Delta_0 + (\Delta_0 - a\,\mathrm{sgn}\,\Delta_0)\cos\omega_n t}$ 1st cycle. valid 'til $\dot{x}=0$ (11)

$\boxed{0 < t < \pi/\omega_n}$

Now make these replacements in (11)

$\Delta_0 :\to x(\frac{\pi}{\omega_n}) = a\,\mathrm{sgn}\,\Delta_0 - \Delta_0 + a\,\mathrm{sgn}\,\Delta_0$ (12)

$\mathrm{sgn}\,\Delta_0 :\to -\mathrm{sgn}\,\Delta_0 \qquad t :\to t - \frac{\pi}{\omega_n}$

Then ...

$$x(t) = -a\,\mathrm{sgn}\,\Delta_0 + (\Delta_0 - 3a\,\mathrm{sgn}\,\Delta_0)\cos\omega_n t \quad (13)$$

for $\pi < \omega_n t < 2\pi$

Carry this process on. By induction,

$\boxed{x(\omega_n t) = (-1)^n a\,\mathrm{sgn}\,\Delta_0 + [\Delta_0 - (2n+1)\,a\,\mathrm{sgn}\,\Delta_0]\cos\omega_n t}$ (14)

$\boxed{n\pi < \omega_n t < (n+1)\pi}$ $\boxed{a = f/k}$ (15)

$x(n\pi) = (-1)^n[\Delta_0 - 2na\,\mathrm{sgn}\,\Delta_0]$ $\qquad n$ = half cycle # $= 0, 1, 2, \cdots$ (16)

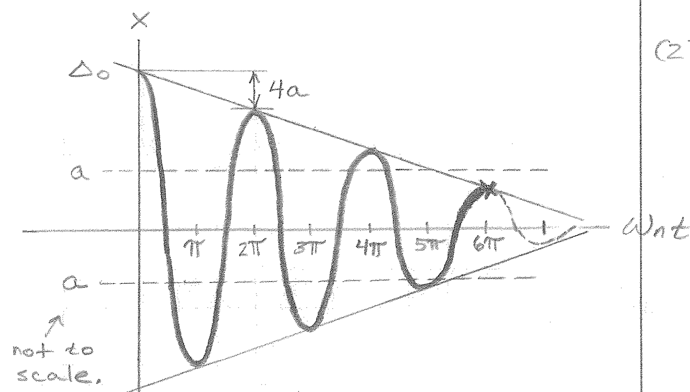
COULOMB (cont'd)

The block will stop when the spring force is not sufficient to overcome the friction force. That is,

BLOCK STOPS when $kx = f$, or $\underset{\substack{\uparrow \\ \text{when } \dot{x}=0}}{x(n\pi)} \leq \pm a$ (1)

Setting $x(n\pi) = \pm a$ in (16) yields

$$\boxed{\begin{array}{l} \#\underline{\text{full}} \text{ cycles to stop} \\ N = 2n = \frac{1}{4}\left(\frac{\Delta_0}{a} - 1\right) \\ \text{ROUND FRACTIONS } \underline{\text{UP}} \text{ TO THE} \\ \text{NEAREST } \underline{\text{HALF}} \text{ CYCLE} \end{array}}$$ (2)



EQUIVALENT VISCOUS DAMPING

Recall (13-3) for forced-viscous systems,

$f(t) = F\cos\Omega t$

$X_p = \bar{X}\cos(\Omega t - \gamma)$ $\quad \gamma = \frac{\pi}{2}$ when $r\left(=\frac{\Omega}{\omega_n}\right) = 1 \quad \forall \zeta$ (3)

Assume
1. When $r = 1$ (resonance), the phase angle $\gamma = \pi/2$ for any kind of (small) nonviscous damping.
2. Steady state forced vibrations are harmonic.

CONCEPT OF EQUIV. VISC. DAMPING:

The equivalent viscous damping factor ζ_{eq} (or C_{eq}) for a non-viscous system preserves amplitude.

Energy input ($r=1$), eq (22-3) $\quad W_{ext} = \pi F \bar{X}$ (4)

Energy dissipation, eq (22-4) $\quad W_{damp} = \pi c \Omega \bar{X}^2 \overset{(13\text{-}3d)}{=} 2\pi r \zeta k \bar{X}^2$ (5)

To get amplitude, $\quad \boxed{\begin{array}{l}(W_d)_{N.V.} = (W_{ext})_{N.V} \\ (W_d)_{N.V.} = (W_d)_{visc.}\end{array}}$ (6)

To get ζ_{eq},

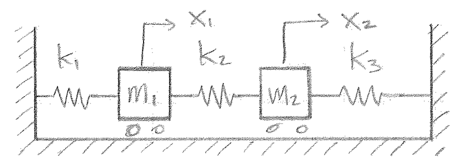
$$W_{damp} = \int_{cycle} f_{damp}\, dx = \int_0^{2\pi/\Omega} f_d \dot{x}\, dt \qquad (7)$$

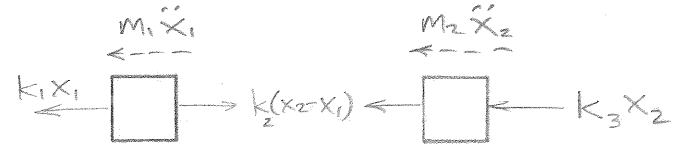
Assuming harmonic vibrations, $x = \bar{X}\cos\Omega t$, so that for $f = f(x, \dot{x}, \ddot{x}, \ldots)$

$$W_{damp} = \int_0^{2\pi} f\left[\bar{X}\cos\Omega t, -\Omega\bar{X}\sin\Omega t, \cdots\right](-\Omega\bar{X}\sin\Omega t)\, d(\Omega t) \qquad (8)$$

$\underline{\text{set}}$ $\pi F \bar{X}$ $\rightarrow$ solve for $\bar{X}$ then $\boxed{\zeta_{eq} = \frac{W_{damp}}{2\pi r k \bar{X}^2}}$ (9)

TWO DOF SYSTEMS





$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2-x_1) &= 0 \\ m_2\ddot{x}_2 + k_3x_2 + k_2(x_2-x_1) &= 0 \end{aligned}\right\} \quad \underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\underline{\underline{M}}}\begin{Bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{Bmatrix} + \underbrace{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}}_{\underline{\underline{K}}}\begin{Bmatrix}x_1 \\ x_2\end{Bmatrix} = \begin{Bmatrix}0 \\ 0\end{Bmatrix} \qquad (1)$$

or $\quad \underline{\underline{M}}\cdot\ddot{\underline{x}} + \underline{\underline{K}}\cdot\underline{x} = \underline{0}$

Assume a sol'n of the form $\underline{x} = \underline{X}\cos(\omega_n t - \gamma)$

$$\Rightarrow \ddot{\underline{x}} = -\omega_n^2\,\underline{x} \qquad (2)$$

$\Rightarrow \quad \left[\underline{\underline{K}} - \omega_n^2\,\underline{\underline{M}}\right]\cdot\underline{X} = \underline{0} \qquad$ Ah! General Eigenproblem!

Case: all k + m. Then

$$m\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\ddot{\underline{x}} + k\begin{bmatrix}2 & -1 \\ -1 & 2\end{bmatrix}\underline{x} = \underline{0} \qquad (3)$$

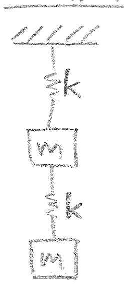
$$\Rightarrow \left\{\begin{bmatrix}2 & -1 \\ -1 & 2\end{bmatrix} - \frac{\omega_n^2}{k/m}\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\right\}\cdot\underline{X} = \underline{0} \qquad (4)$$

$$\Rightarrow \frac{\omega_n^2}{k/m} = 1,\ 3 \qquad \underline{X} = \begin{Bmatrix}1 \\ 1\end{Bmatrix},\ \begin{Bmatrix}1 \\ -1\end{Bmatrix} \qquad (5)$$

$$\underline{x} = A_1\,\underline{X}_1 + A_2\,\underline{X}_2 \quad \text{or} \quad \begin{aligned} x_1 &= A_1\cos(\omega_1 t - \gamma_1) + A_2\cos(\omega_2 t - \gamma_2) \\ x_2 &= A_1\cos(\omega_1 t - \gamma_1) - A_2\cos(\omega_2 t - \gamma_2) \end{aligned} \qquad (6)$$

$A_1, A_2, \gamma_1, \gamma_2$ come from IC's

EXAMPLE



$$\underline{\underline{K}} = k\begin{bmatrix}2 & -1 \\ -1 & 1\end{bmatrix} \qquad \underline{\underline{M}} = m\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} \qquad (7)$$

$$\Rightarrow \frac{\omega_n^2}{k/m} = \tfrac{1}{2}(3 \mp \sqrt{5}) = \begin{cases}0.3820 \\ 2.6180\end{cases} \qquad (8)$$

$$\underline{X} = \begin{Bmatrix}-1 \pm \sqrt{5} \\ 2\end{Bmatrix} \qquad (9)$$

ratio of amplitudes:

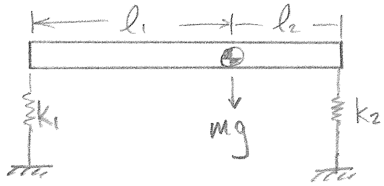
$$\begin{cases}0.618 \\ -1.618\end{cases} \qquad (10)$$

GENERAL COORDINATE COUPLING

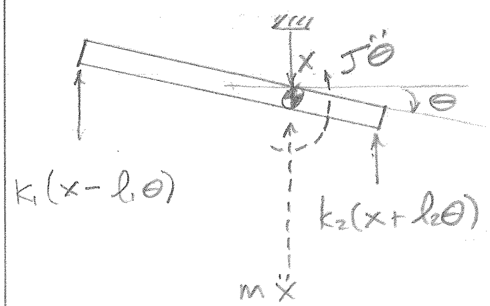
$$\underline{M}\,\ddot{\underline{x}} + \underline{K}\,\underline{x} = \underline{0} \qquad (1)$$

"Mass" or "Dynamic" coupling $\Rightarrow$ $\underline{M}$ non-diag
"Stiffness" or "static" coupling $\Rightarrow$ $\underline{K}$ non-diag

EXAMPLE Off center center of mass



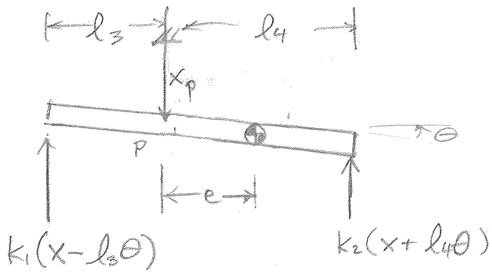
STATIC COUPLING:



$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & k_2l_2 - k_1l_1 \\ k_2l_2 - k_1l_1 & k_2l_2^2 + k_1l_1^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (2)$$

Uncoupled if $k_2l_2 = k_1l_1$

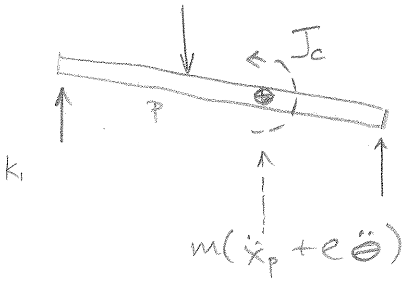
DYNAMIC COUPLING:
Select x above the point where a vertical force produces pure translation, that is, where $l_3 k_1 = l_4 k_2$



$$\begin{bmatrix} m & me \\ me & J_p \end{bmatrix}\begin{Bmatrix} \ddot{x}_p \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & 0 \\ 0 & k_2l_4^2 + k_1l_3^2 \end{bmatrix}\begin{Bmatrix} x_p \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3)$$

Two ways to analyze:

Delambert



Basic eqn w/ noninertial or cm reference:

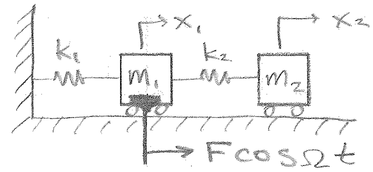
$$\underline{M}_p - \underbrace{\underline{\rho}_c \times m\,\ddot{\underline{r}}_p}_{(e)(-m\ddot{x})} = \dot{\underline{H}}_p = -J_p\ddot{\theta}$$

$$\Rightarrow \quad \underline{M}_p + me\ddot{x}_p + J_p\ddot{\theta} = 0$$

same! ✓

$$\Rightarrow \underline{M}_p = \underline{M}_p + me\ddot{x}_p + \underbrace{(J_c + me^2)}_{J_p}\ddot{\theta} = 0$$

HARMONICALLY FORCED - UNDAMPED 2DOF SYSTEMS



$$m_1\ddot{x}_1 - k_2(x_2-x_1) + k_1x_1 = F\cos\Omega t$$
$$m_2\ddot{x}_2 + k_2(x_2-x_1) = 0$$

In matrix form $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}\cos\Omega t$

For the particular solution, try

$$\{x_p\} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}\cos\Omega t$$

Then $-\Omega^2 \underline{M}\cdot\underline{X} + \underline{K}\cdot\underline{X} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$

Notice: this is not an eigenproblem because $F \neq 0$.

The sol'n for X is ...

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \frac{F}{D}\begin{Bmatrix} k_2 - m_2\Omega^2 \\ k_2 \end{Bmatrix} \quad \text{where } D = (k_1+k_2-m_1\Omega^2)(k_2-m_2\Omega^2) - k_2^2$$

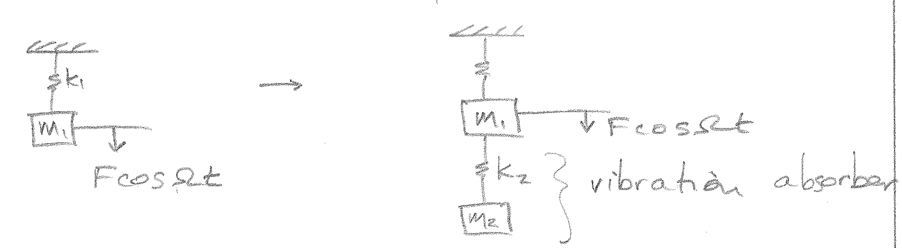
CASE: $\Omega \neq \omega_n$

1) X_1 & X_2 $\propto F$
2) X_2 can never be zero
3) X_1 can be zero if $\Omega = \sqrt{\frac{k_2}{m_2}}$

V I B R A T I O N A B S O R B E R

SUB-CASE $\Omega = \sqrt{k_2/m_2}$

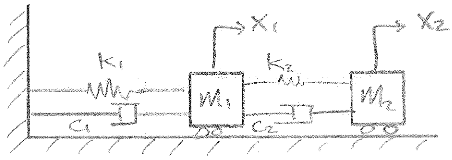
1) $X_1 \equiv 0$
2) This is very useful - can design a vibration absorber when Ω is near ω_n of the SDOF system k_1m_1



Chose $\sqrt{\frac{k_1}{m_1}} = \Omega$ to get zero disp of m_1 and pick k_1 & m_1 individually according to allowable value of X_2.

Disadvantage: must pass thru. resonance of X_2 to arrive at Ω.

2 DOF FORCED W/ DAMPING



$\mapsto F e^{i\Omega t}$

EOM:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} e^{i\Omega t} \qquad (1)$$

Particular soln $\quad x = X e^{i(\Omega t - \alpha)} = \underbrace{X e^{-i\alpha}}_{\bar{X}} e^{i\Omega t} = \bar{X} e^{i\omega t} \qquad (2)$

$\bar{X}$ cmpx ampl. incl. phase!

Subst. in to get

$$\begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = \frac{F}{D} \begin{Bmatrix} k_2 - m\Omega^2 + i\Omega c_2 \\ k_2 + i\Omega c_2 \end{Bmatrix} = \begin{Bmatrix} X_1 e^{-i\alpha_1} \\ X_2 e^{-i\alpha_2} \end{Bmatrix} \qquad (3)$$

$$D = \left[(k_1+k_2 - m_1\Omega^2) + i\Omega(c_1+c_2)\right](k_2 - m_2\Omega^2 + i\Omega c_2) - (k_2 + i\omega c_2)^2 \qquad (4)$$

EIGENVALUE ANALYSIS OF UNDAMPED FREE SYSTEM

EOM:

$$\underline{\underline{M}} \cdot \ddot{\underline{x}} + \underline{\underline{K}} \cdot \underline{x} = \underline{0} \qquad (5)$$

"DYNAMIC MATRIX"

$$\underline{\underline{M}}^{-1} \| \Rightarrow \quad \underline{\underline{I}} \cdot \ddot{\underline{x}} + \underline{\underline{A}} \cdot \underline{x} = 0 \qquad \boxed{\underline{\underline{A}} \equiv \underline{\underline{M}}^{-1} \cdot \underline{\underline{K}}} \qquad (6)$$

Eigen problem

$$[\underline{\underline{A}} - \lambda \underline{\underline{I}}] \cdot \underline{x} = 0 \quad \text{or} \quad [\underline{\underline{K}} - \lambda \underline{\underline{M}}] \cdot \underline{x} = 0 \qquad (7)$$

Orthogonality

$$\underline{x}_i \cdot \underline{\underline{M}} \cdot \underline{x}_j = \begin{cases} 0 & \text{if } i \neq j \\ M_i & \text{if } i = j \end{cases} \quad \text{"generalized mass"} \qquad (8)$$

$$\underline{x}_i \cdot \underline{\underline{K}} \cdot \underline{x}_j = \begin{cases} 0 & \text{if } i \neq j \\ K_i & \text{if } i = j \end{cases} \quad \text{"generalized stiffness"} \qquad (9)$$

Modal matrix

$$\underline{\underline{P}} = \left[\underline{x}_1 \, \underline{x}_2 \cdots \underline{x}_n \right]_{n \times n} \qquad (10)$$

$$\underline{\underline{P}}^T \cdot \underline{\underline{M}} \cdot \underline{\underline{P}} = \left[{}^{\diagdown} M_i {}_{\diagdown} \right] \qquad (11)$$

$$\underline{\underline{P}}^T \cdot \underline{\underline{K}} \cdot \underline{\underline{P}} = \left[{}^{\diagdown} K_i {}_{\diagdown} \right] \qquad (12)$$

Weighted modal matrix

$$\tilde{P} = \left[\frac{x_1}{\sqrt{M_1}} \quad \frac{x_2}{\sqrt{M_2}} \quad \cdots \quad \frac{x_n}{\sqrt{M_n}} \right] \tag{1}$$

$$\underline{\underline{\tilde{P}}}^T \cdot \underline{\underline{M}} \cdot \underline{\underline{\tilde{P}}} = \underline{\underline{I}} \tag{2}$$

$$\underline{\underline{\tilde{P}}}^T \cdot \underline{\underline{K}} \cdot \underline{\underline{\tilde{P}}} = \underline{\underline{\Lambda}} = \begin{bmatrix} \ddots & & \\ & \lambda & \\ & & \ddots \end{bmatrix} \tag{3}$$

Principal coordinates.

Define $\underline{y}$ by

PRINCIPAL COORDINATES

$$x = \underline{\underline{\tilde{P}}} \cdot \underline{y} \quad \Rightarrow \quad \boxed{\underline{y} = \underline{\underline{\tilde{P}}}^{-1} \cdot x = \underline{\underline{\tilde{P}}}^T \underline{\underline{M}} \; \underline{x}} \tag{4}$$

Uncoupling the governing ODE:

$$\text{ODE:} \quad \underline{\underline{M}} \cdot \ddot{\underline{x}} + \underline{\underline{K}} \cdot \underline{x} = \underline{0} \tag{5}$$

$$(4a) \rightarrow \underline{\underline{\tilde{P}}}^T \Big\| \quad \underline{\underline{M}} \cdot (\underline{\underline{\tilde{P}}} \ddot{\underline{y}}) + \underline{\underline{K}} \cdot (\underline{\underline{\tilde{P}}} \cdot \underline{y}) = \underline{0} \tag{6}$$

$$(2) \Rightarrow \quad I \, \ddot{\underline{y}} + \underline{\underline{\Lambda}} \, \underline{y} = \underline{0} \qquad \textit{Ah! Uncoupled!} \tag{7}$$

INFLUENCE COEFFICIENTS

Flexibility α_{ij} = disp at i due to unit force applied at j (8)

Stiffness k_{ij} = force required at i due to unit disp at j (9)

Reciprocity thm

$$\begin{aligned} \text{Work}_{ij} &= \tfrac{1}{2} f_i (\alpha_{ii} f_i) + \tfrac{1}{2} f_j (\alpha_{jj} f_j) + f_i (\alpha_{ij}) f_j \\ \text{Work}_{ji} &= \tfrac{1}{2} f_j (\alpha_{jj} f_j) + \tfrac{1}{2} f_i (\alpha_{ii} f_i) + f_j (\alpha_{ji}) f_i \end{aligned} \tag{10}$$

must be equal $\Rightarrow \alpha_{ij}$

$$\boxed{\begin{aligned} \alpha_{ij} &= \alpha_{ji}, \\ k_{ij} &= k_{ji} \end{aligned}} \qquad \boxed{\underline{\underline{\alpha}} = \underline{\underline{k}}^{-1}} \tag{11}$$

USING INFLUENCE

$x_i = \sum_j \alpha_{ij} f_j$ where $\underline{f}$ is the vector of applied forces (12)

N's law: f = the Delombert pseudo forces

$$\Rightarrow f_j = -m_j \ddot{x}_j \quad \cancel{\Sigma} j$$

$$\text{EOM:} \quad x_i = -\sum_j \alpha_{ij} m_j \ddot{x}_j$$

$$\text{or} \quad \sum_j (\alpha_{ij} m_j \ddot{x}_j) + x_i = 0$$

In matrix form, the EOM is

$$\{x\} = [\alpha]\{f\} = [\alpha]\{-m_i\ddot{x}_i\} = -[\alpha]\lceil m \rfloor\{\ddot{x}\} = +\omega^2[\alpha]\lceil m \rfloor\{x\} \qquad (1)$$

Similarly, for the stiffness influence,

$$\left.\begin{aligned} \{f\} &= [k]\{x\} \\ \{f\} &= \{-m_i\ddot{x}\} = +\omega^2\lceil m \rfloor\{x\} \end{aligned}\right\} \Rightarrow \{x\} = \frac{1}{\omega^2}\lceil m \rfloor^{-1}[k]\{x\} \qquad (2)$$

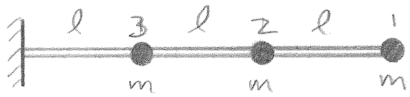
Summary

$$[k] = [\alpha]^{-1} \qquad \boxed{\begin{aligned} \{x\} &= \omega^2[\alpha]\lceil m \rfloor\{x\} \\ \{x\} &= \frac{1}{\omega^2}\lceil m \rfloor^{-1}[k]\{x\} \end{aligned}} \quad \begin{aligned} &\text{dynamic coupling} \\ &\text{static coupling} \end{aligned} \qquad (3)$$

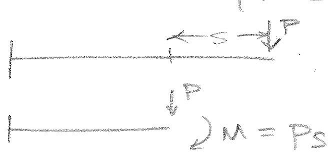
or $\quad \lceil m \rfloor\{\ddot{x}\} + [k]\{x\} = 0 \qquad (4)$

MATRIX ITERATION

EX: Find the natural freq + mode shapes for the lumped cantilever beam:



Need an expression for the disp due to a unit load



$$\delta = \frac{PL^3}{3EI} + \frac{ML^2}{2EI} \qquad (5)$$

since $M = Ps$,

$$\frac{\delta}{P} = \frac{L^3}{3EI} + \frac{sL^2}{2EI} = \frac{L^3}{3EI}\left(1 + \frac{3}{2}\frac{s}{L}\right) \qquad (6)$$

α_{ij} = disp at i due to unit force at j

$$s = 0: \quad \alpha_{11} = \frac{(3l)^3}{3EI} \qquad \alpha_{22} = \frac{(2l)^3}{3EI} \qquad \alpha_{33} = \frac{(l)^3}{3EI}$$

$$\alpha_{12} = \alpha_{21} = \frac{(2l)^3}{3EI}\left(1 + \frac{3}{2}\frac{1}{2}\right) = 14\frac{l^3}{3EI}$$

$$\alpha_{13} = \alpha_{31} = \frac{l^3}{3EI}\left(1 + \frac{3}{2}\frac{2}{1}\right) = 4\frac{l^3}{3EI}$$

$$\alpha_{23} = \alpha_{32} = \frac{l^3}{3EI}\left(1 + \frac{3}{2}\frac{1}{1}\right) = 2.5\frac{l^3}{3EI}$$

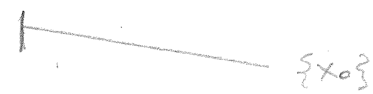
$$\Rightarrow \quad [\alpha] = \frac{l^3}{3EI}\begin{bmatrix} 27 & 14 & 4 \\ & 8 & 2.5 \\ \text{sym.} & & 1 \end{bmatrix} \qquad (7)$$

Now iteration can be set up. From 29-3,

$$\{x\} = \omega_n^2 [\alpha][m]\{x\}$$

since $m = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$,

$$\{x\} = A \begin{bmatrix} 27 & 14 & 4 \\ & 8 & 2.5 \\ \text{sym} & & 1 \end{bmatrix}\{x\} \qquad A = \frac{\omega_n^2 l^3 m}{3EI}$$



As a first guess, take $\{x\}_0 = \begin{Bmatrix} 3 \\ 2 \\ 1 \end{Bmatrix} A$

$$\{x\}_1 = \begin{Bmatrix} 113 \\ 60.5 \\ 18 \end{Bmatrix} A = \begin{Bmatrix} 6.2778 \\ 3.3611 \\ 1 \end{Bmatrix}(18A)$$

$$\{x\}_2 = \begin{Bmatrix} 65.61 \\ 34.89 \\ 10.26 \end{Bmatrix}(18A) = \begin{Bmatrix} 6.3903 \\ 3.3980 \\ 1 \end{Bmatrix}(10.26 \times 18A)$$

convergence

$$\{X\}_3 = \begin{Bmatrix} 224.111 \\ 119.1487 \\ 35.0563 \end{Bmatrix}\begin{pmatrix} 10.26 \\ \times 18A \end{pmatrix} = \begin{Bmatrix} 6.3929 \\ 3.3988 \\ 1 \end{Bmatrix}(35.0563 \times 10.26 \times 18A)$$

$$= 35.0563 \{X_2\}$$

$$\Rightarrow \quad A \approx \frac{1}{35.0563}$$

$$\Rightarrow \quad \boxed{\omega_1^2 \approx \frac{3EI}{ml^3(35.0563)}} \quad \text{and} \quad \boxed{\{X_1\} = \begin{Bmatrix} 6.3929 \\ 3.3988 \\ 1 \end{Bmatrix}}$$

To find higher modes write

$$\underline{x} = C_1 \underline{x}_1 + C_2 \underline{x}_2 + C_3 \underline{x}_3$$

To remove the first mode, require $c_1 = 0$ and premultiply by $\underline{x} \cdot 1$

RAYLEIGH METHOD

By setting $U_{max} = T_{max}$, we can obtain an eqn for the natural frequency

EX: BEAM $\quad u(x,t) = y(x)\,\tau(t) \qquad \tau = A\cos\omega t$

$$U_{max} = \frac{1}{2}\int_0^\ell \frac{M^2}{EI}\,dx = \frac{1}{2}\int_0^\ell EI\,(u'')^2_{max}\,dx = \frac{1}{2}\left(\tau(t)_{max}\right)\int_0^\ell EI\,(y'')^2\,dx$$

$$T_{max} = \frac{1}{2}\int_0^\ell \dot{u}^2\,\bar{\rho}\,dx = \frac{1}{2}\omega^2\tau_{max}\int_0^\ell y^2\,\bar{\rho}\,dx$$

Since $\tau(t)_{max} = 1$,

$$U_{max} = \frac{1}{2}\int_0^\ell EI\,(y'')^2\,dx$$

$$T_{max} = \frac{1}{2}\omega^2\int_0^\ell y^2\,\bar{\rho}\,dx$$

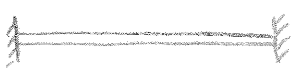
By equating energies $T_{max} = U_{max}$, find

$$\boxed{\omega^2 = \frac{\int_0^\ell EI\,(y'')^2\,dx}{\int_0^\ell y^2\,\bar{\rho}\,dx} = \frac{EI}{\bar{\rho}\,\ell^4}\,\frac{\int_0^1 (y_{,\eta\eta})^2\,d\eta}{\int_0^1 y^2\,d\eta}}$$

$\int_0^1 (y_{,\eta\eta})^2\,d\eta = \int y\,y_{,\eta\eta\eta\eta}\,d\eta$?

where $y(\eta)$ is a nondimensional admissible approximation to the deflection.

EXAMPLE



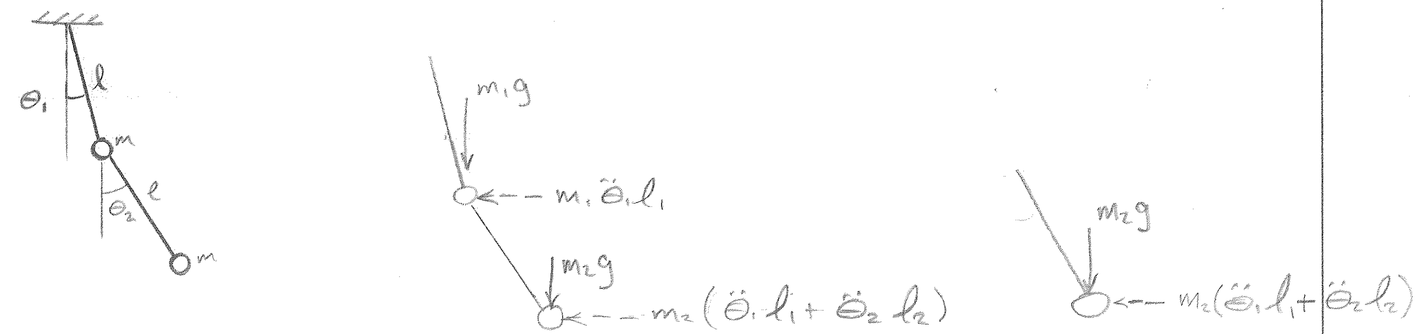
try $\quad y = \eta^2(1-\eta)^2 \qquad y_{,\eta\eta} = 2(1 - 6\eta + 6\eta^2)$

$$(y_{,\eta\eta})^2 = 4\left[1 - 12\eta + 48\eta^2 - 72\eta^3 + 36\eta^4\right]$$

$$\frac{\int_0^1 (y_{,\eta\eta})^2\,d\eta}{\int_0^1 y^2\,d\eta} = \frac{4/5}{1/630} = 504$$

$$\Rightarrow \quad \omega^2 = 504\,\frac{EI}{\bar{\rho}\,\ell^4} \qquad 0.34\%\ \text{error!}$$

DOUBLE PENDULUM



$$m_1 g\, l_1\theta_1 + m_2 g\,(l_1\theta_1 + l_2\theta_2) + m_1\ddot{\theta}_1 l_1^2 + m_2(\ddot{\theta}_1 l_1 + \ddot{\theta}_2 l_2)(l_1 + l_2) = 0$$

$$m_2 g\, l_2\theta_2 + m_2\ddot{\theta}_1 l_1 l_2 + m_2\ddot{\theta}_2 l_2^2 = 0$$

If $l_1 = l_2 = l \quad m_1 = m_2 = m$

$$l\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{Bmatrix} + g\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = 0$$

EVALS:

$$\det\left|\begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix} - \frac{\omega_n^2}{g/l}\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}\right| = 0$$

$$\Rightarrow \quad \boxed{\omega_n^2 = (2 \pm \sqrt{2})\frac{g}{l}}$$

another form for the governing eqns

$$l\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{Bmatrix} + g\begin{bmatrix}2 & -1\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

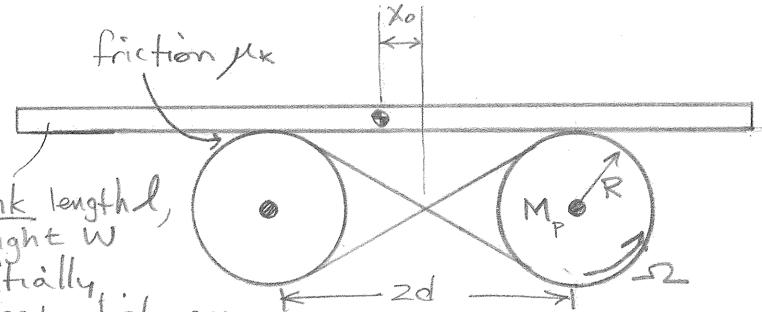
Evecs

$$\begin{Bmatrix}\hat{\theta}\end{Bmatrix} = \begin{Bmatrix}1-\lambda\\ +\lambda\end{Bmatrix} = \begin{Bmatrix}-1 \mp \sqrt{2}\\ 2 \pm \sqrt{2}\end{Bmatrix} = \begin{Bmatrix}\mp 1/\sqrt{2}\\ 1\end{Bmatrix} = \begin{Bmatrix}1\\ \mp\sqrt{2}\end{Bmatrix}$$

Appendix

GIVEN:

A plank is placed on two counter rotating pulleys in the initial configuration shown. The mass of each pulley is M_p. The pulleys are initially set to rotate at an angular frequency Ω and then are free.

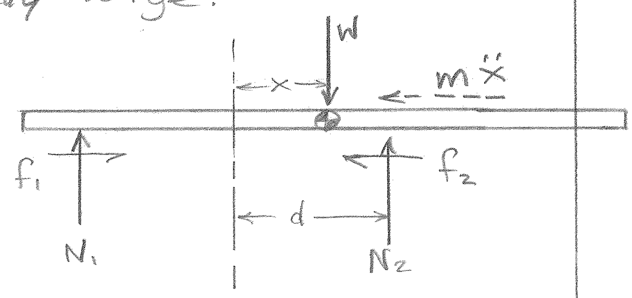


Plank length l, weight W initially offset distance x_0 from center of pulleys.

REQ'D How long will it take for the system to come to a rest assuming Ω is initially large?

SOL'N

Let $x =$ offset



$$N_1 = \frac{d-x}{2d} W \qquad f_1 = \mu N_1$$

$$N_2 = \frac{d+x}{2d} W \qquad f_2 = \mu N_2$$

$$\boxed{\Sigma F = m\ddot{x}}$$

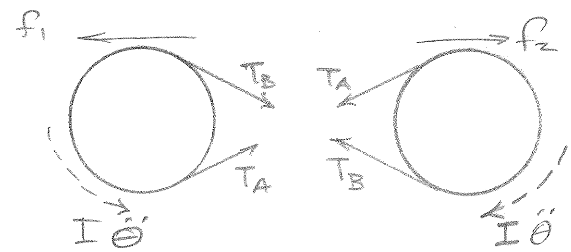
$$m\ddot{x} + f_2 - f_1 = 0$$

$$\boxed{m\ddot{x} + \frac{\mu W}{d} x = 0}$$

Harmonic motion! $\omega_n = \sqrt{\frac{\mu W}{md}} = \sqrt{\frac{\mu g}{d}}$

$$\boxed{x = x_0 \cos \omega_n t}$$

Pulleys



$$(f_1 + T_A - T_B)R + I\ddot{\theta} = 0$$

$$(f_2 - T_A + T_B)R + I\ddot{\theta} = 0$$

$$R(f_1 + f_2) + 2I\ddot{\theta} = 0$$

$$\ddot{\theta} = \frac{R(f_1+f_2)}{2I} = \frac{\mu R W}{2(\frac{1}{2}MR^2)} = \frac{\mu g}{R}\frac{m}{M}$$

$$\Rightarrow t = \frac{\Omega R}{\mu g}\frac{M}{m}$$

each other. One can orthogonalize them by taking them in certain linear combinations.

When $r = s$ the triple products in (4.52) and (4.53) are not zero, but yield mass and stiffness coefficients

$$M_{rr} = \{u^{(r)}\}^T[m]\{u^{(r)}\}, \tag{4.54}$$

$$K_{rr} = \{u^{(r)}\}^T[k]\{u^{(r)}\}. \tag{4.55}$$

Equations (4.52) and (4.54), on the one hand, and (4.53) and (4.55), on the other, yield

$$[u]^T[m][u] = \lceil M \rfloor, \tag{4.56}$$

$$[u]^T[k][u] = \lceil K \rfloor, \tag{4.57}$$

and if the eigenvectors $\{u^{(r)}\}$ are normalized according to (4.17), the above reduce to

$$[u]^T[m][u] = \lceil M \rfloor = [I], \tag{4.58}$$

$$[u]^T[k][u] = \lceil K \rfloor = \lceil \omega^2 \rfloor. \tag{4.59}$$

The orthogonality property has an important implication. Consider a set of n linear independent vectors in an n-dimensional space. Such a set is called a *basis* in that space. Any vector in the n-dimensional space can be generated by a linear combination of the linearly independent vectors. This fact implies solving a set of n algebraic equations, the coefficients of which are the elements of the linearly independent vectors. Hence the determinant of the coefficients cannot vanish and a nontrivial solution exists. It follows that the set of characteristic vectors $\{u^{(1)}\}, \{u^{(2)}\}, \ldots, \{u^{(n)}\}$, because it is orthogonal, hence independent, is a complete set of vectors in the sense that it can be used as a basis for the decomposition of any arbitrary n-dimensional vector $\{x\}$. This statement is known as the *expansion theorem* and can be expressed

$$\{x\} = \sum_{r=1}^{n} c_r\{u^{(r)}\}, \tag{4.60}$$

where the coefficients c_r are given by

$$c_r = \{u^{(r)}\}^T[m]\{x\}, \qquad r = 1, 2, \ldots, n. \tag{4.61}$$

The expansion theorem is of extreme importance in the field of vibrations and forms the basis of a widely used method of obtaining the system response known as *modal analysis*. Modal analysis will be used extensively throughout this text and will be discussed in detail in Chapter 7.

4-6 MATRIX ITERATION METHOD. SWEEPING TECHNIQUE

The method described in Section 4–3 involves the expansion of the determinantal equation and obtaining the roots of the frequency equation, which in the case of an n-degree-of-freedom system is an algebraic equation of nth order. A different approach uses matrix iteration procedures. We shall discuss two matrix iteration procedures: the first one iterates to one mode at a time and is known as the *matrix iteration method*, and the second one iterates to all modes simultaneously and we shall call it the *diagonalization by successive rotations method*. The first procedure will be discussed in this section.

The matrix iteration method is based on the assumption that the natural frequencies are distinct: $\omega_1 < \omega_2 < \omega_3 < \cdots < \omega_n$. The rate of convergence is deeply affected if two natural frequencies have very close values.

Equation (4.31) leads to the eigenvalue problem in the form

$$[D]\{u\} = \frac{1}{\omega^2}\{u\}, \tag{4.62}$$

which is satisfied by every eigenvector $\{u^{(r)}\}$ with corresponding natural frequency ω_r.

The premultiplication of an arbitrary vector $\{u\}_1$ by the matrix $[D]$ represents a linear transformation that transforms the trial vector $\{u\}_1$ into another vector $\{u\}_2$. If the vector $\{u\}_1$ is one of the eigenvectors, say $\{u^{(r)}\}$, premultiplication by $[D]$ results in a vector $\{u\}_2$ which is proportional to $\{u^{(r)}\}$. If the trial vector $\{u\}_1$ is not an eigenvector, the resulting vector $\{u\}_2$ can be used as an improved trial vector. A sequence of such linear transformations eventually leads to a vector which when premultiplied by $[D]$ transforms into a vector proportional to itself. At this point convergence has been achieved, and the vector is simply the first eigenvector $\{u^{(1)}\}$ and the constant of proportionality is ω_1^{-2}. The following discussion justifies these statements.

According to the *expansion theorem* the n orthogonal eigenvectors $\{u^{(r)}\}$ of a system can be used as a basis for the decomposition of any arbitrary n-dimensional vector $\{u\}$ representing a possible motion of the system. Denote this vector $\{u\}_1$, such that

$$\{u\}_1 = c_1\{u^{(1)}\} + c_2\{u^{(2)}\} + \cdots + c_n\{u^{(n)}\} = \sum_{r=1}^{n} c_r\{u^{(r)}\}, \tag{4.63}$$

where the constants c_r play the role of the unknowns in the set of n algebraic equations.

Any vector $\{u^{(r)}\}$ premultiplied by the dynamical matrix $[D]$ reproduces itself. Hence, if we use $\{u\}_1$, as given by (4.63), as a trial vector, premultiplication by $[D]$ yields

$$\{u\}_2 = [D]\{u\}_1 = \sum_{r=1}^{n} c_r[D]\{u^{(r)}\} = \sum_{r=1}^{n} \frac{c_r}{\omega_r^2}\{u^{(r)}\}. \tag{4.64}$$

Next premultiply $\{u\}_2$ by $[D]$ and call the result $\{u\}_3$, to obtain

$$\{u\}_3 = [D]\{u\}_2 = \sum_{r=1}^{n} \frac{c_r}{(\omega_r^2)^2}\{u^{(r)}\}. \tag{4.65}$$

In general, we have

$$\{u\}_p = [D]\{u\}_{p-1} = \sum_{r=1}^{n} \frac{c_r}{(\omega_r^2)^{p-1}}\{u^{(r)}\}. \tag{4.66}$$

But the natural frequencies are such that $\omega_1 < \omega_2 < \cdots < \omega_n$. As p increases indefinitely, the first term of the series in (4.66) becomes predominantly larger than the other terms and in the limit, as $p \to \infty$, the trial vector $\{u\}_p$ will resemble in shape the first eigenvector $\{u^{(1)}\}$. Hence we have

$$\lim_{p\to\infty} \{u\}_p = \{u^{(1)}\}, \tag{4.67}$$

and the first natural frequency is obtained from

$$\lim_{p\to\infty} \frac{u_{i,p-1}}{u_{i,p}} = \omega_1^2, \tag{4.68}$$

where $u_{i,p-1}$ and $u_{i,p}$ are the elements in the ith row of the trial vector $\{u\}_{p-1}$ and the resulting vector $\{u\}_p$, respectively. Of course, in practice a finite number of iterations is sufficient for a good estimate of the first mode, the number of iterations depending on the accuracy desired.

For a given desired degree of accuracy there are two factors affecting the number of iterations necessary. The first one is how closely the arbitrary trial vector $\{u\}_1$ resembles the first mode. Mathematically this is equivalent to asking how large the coefficient c_1 is compared to the other $n - 1$ coefficients, if the eigenvectors $\{u^{(r)}\}$ are regarded as normalized. Obviously if c_1 is large compared to the remaining coefficients, the trial vector resembles the first mode to some degree and a smaller number of iterations will be necessary. This factor depends on the skill and experience of the analyst. The second factor depends entirely on the system, and it concerns the relative values of ω_1 and ω_2. The larger ω_2 is compared to ω_1, the faster the modes will separate and the smaller the necessary number of iterations will be. It is obvious that this method converges to the lowest mode.

One advantage of this method is that computational errors do not bring wrong results. Any error in one of the premultiplications by $[D]$ does not have a persistent damaging effect, because one can look upon the vector in error as a new trial vector. Errors, in general, will delay the convergence. Any set of numbers can be chosen for the trial vector $\{u\}_1$. Even if the numbers are such that the coefficient c_1 is almost zero, convergence will be achieved, although it may take longer. If after one iteration cycle the values of the elements of the resulting vector are too large, one can scale them down proportionately, because this does not affect the shape of the vector. Only in the unusual case in which the trial vector $\{u\}_1$ is exactly proportional to one of the modes other than the first mode, say $\{u^{(s)}\}$, does the method fail to deliver the first mode, because premultiplication by $[D]$ transforms $\{u^{(s)}\}$ into itself.

The method described above gives us the first or fundamental mode. The question remains how to obtain the higher modes. Any arbitrary trial vector premultiplied by $[D]$ would lead again to the first mode, so we must modify the procedure to obtain the second mode. To this end we must insist that the trial vector for the second mode is independent of the first eigenvector. That means the trial vector for the second mode must be orthogonal to the eigenvector $\{u^{(1)}\}$. The orthogonality relation is expressed by

$$\{u\}^T[m]\{u^{(1)}\} = 0. \tag{4.69}$$

Now introduce the notation

$$\{m^{(r)}\} = [m]\{u^{(r)}\}, \qquad r = 1, 2, \ldots, n. \tag{4.70}$$

so the orthogonality relation takes the form of the constraint equation

$$u_1 m_1^{(1)} + u_2 m_2^{(1)} + \cdots + u_n m_n^{(1)} = 0. \tag{4.71}$$

Equation (4.71) can be solved for one of the variables, say u_1, in terms of the remaining $n - 1$ variables. Letting

$$\frac{m_i^{(1)}}{m_1^{(1)}} = m_{1i}^{(1)}, \tag{4.72}$$

we obtain

$$u_1 = -m_{12}^{(1)}u_2 - m_{13}^{(1)}u_3 - \cdots - m_{1n}^{(1)}u_n, \tag{4.73}$$

and we may choose the remaining $n - 1$ variables arbitrarily,

$$u_i = u_i, \qquad i = 2, 3, \ldots, n. \tag{4.74}$$

Next construct a constraint matrix

$$[S^{(1)}] = \begin{bmatrix} 0 & -m_{12}^{(1)} & -m_{13}^{(1)} & \cdots & -m_{1n}^{(1)} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{4.75}$$

called a *sweeping matrix*, which allows us to write (4.73) and (4.74) in the compact form

$$\{u\}_c^{(2)} = [S^{(1)}]\{u\}, \tag{4.76}$$

where $\{u\}$ indicates an arbitrary vector and $\{u\}_c^{(2)}$ denotes a constrained vector orthogonal to the eigenvector $\{u^{(1)}\}$. At every stage of the iteration for the second mode we must be sure that the first mode is suppressed. This leads to the matrix iteration for the second mode in the form

$$[D]\{u\}_c^{(2)} = [D][S^{(1)}]\{u\}, \tag{4.77}$$

so it is convenient to devise a new dynamical matrix,

$$[D^{(2)}] = [D][S^{(1)}], \tag{4.78}$$

which has all the elements in the first column zero. The matrix $[D^{(2)}]$ leads to convergence to the second mode in the same way $[D]$ brings about convergence to the first. The eigenvalue problem now has the form

$$[D^{(2)}]\{u\} = \frac{1}{\omega^2}\{u\}, \tag{4.79}$$

and the iteration to the second mode follows the same pattern as the one for the first mode. An arbitrary trial vector is selected and premultiplication by $[D^{(2)}]$ yields an improved trial vector which is in turn premultiplied by $[D^{(2)}]$. We can speed up the convergence by choosing a trial vector reasonably close to the second mode. This is not always possible nor is it necessary. Here, too, the method fails to furnish the second mode if the arbitrary trial vector happens to coincide with one of the modes other than the second one.

To obtain the third mode we must insist that both the first and second modes are suppressed from the trial vectors. Hence the trial vector must be orthogonal to both first and second modes, and the orthogonality conditions are

$$\{u\}^T[m]\{u^{(1)}\} = 0, \tag{4.80}$$

$$\{u\}^T[m]\{u^{(2)}\} = 0, \tag{4.81}$$

which represent two simultaneous constraint equations in the unknowns $u_1, u_2, \ldots, u_n$. Solving for u_1 and u_2 in terms of the remaining $n - 2$ unknowns and arbitrarily letting $u_i = u_i$, for $i = 3, 4, \ldots, n$, we can construct a new sweeping matrix $[S^{(2)}]$ in the form

$$[S^{(2)}] = \begin{bmatrix} 0 & 0 & m_{23}^{(2)} & m_{24}^{(2)} & \cdots & m_{2n}^{(2)} \\ 0 & 0 & -m_{13}^{(2)} & -m_{14}^{(2)} & \cdots & -m_{1n}^{(2)} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{4.82}$$

where $$m_{ij}^{(2)} = \frac{m_i^{(2)}m_j^{(1)} - m_i^{(1)}m_j^{(2)}}{m_1^{(2)}m_2^{(1)} - m_1^{(1)}m_2^{(2)}}, \qquad i = 1, 2; j = 3, 4, \ldots, n. \tag{4.83}$$

Correspondingly we obtain a third dynamical matrix,

$$[D^{(3)}] = [D][S^{(2)}], \tag{4.84}$$

having the elements in both first and second columns zero, so the eigenvalue problem

$$[D^{(3)}]\{u\} = \frac{1}{\omega^2}\{u\} \tag{4.85}$$

yields the third mode.

The procedure follows the same pattern for the remaining modes, which are all obtained in ascending order. This is a numerical procedure, so accuracy is lost as the mode sought becomes higher. It is highly desired that at the beginning a larger number of significant figures than the one desired for the lower modes is retained, thus improving the accuracy of the higher modes.

The eigenvalue problem as given by (4.62) can also be written

$$[D]^{-1}\{u\} = \omega^2\{u\}, \tag{4.86}$$

and it follows that one can converge to the highest mode $\{u^{(n)}\}$ by premultiplying the trial vectors $\{u\}_p$ by $[D]^{-1}$ instead of $[D]$. Otherwise the method of solution is similar to the one already described. Aside from the fact that the modes are obtained in descending order, the constant of proportionality is in this case ω^2 rather than ω^{-2}.

It should be noted that to obtain the last mode it is no longer necessary to iterate, because the $n - 1$ orthogonality conditions are sufficient to define the last mode. One may wish, however, to follow the iteration procedure as a means of obtaining the last eigenvalue.

Example 4.2

Solve the eigenvalue problem of Example 4.1 by the matrix iteration method, using the sweeping technique.

In Example 4.1 we obtained the dynamical matrix

$$[D] = \frac{LI_D}{7GJ}\begin{bmatrix} 5 & 3 & 2 \\ 3 & 6 & 4 \\ 1 & 2 & 6 \end{bmatrix}, \tag{a}$$

so introducing the above matrix $[D]$ in (4.62) we obtain the following eigenvalue problem:

$$\begin{bmatrix} 5 & 3 & 2 \\ 3 & 6 & 4 \\ 1 & 2 & 6 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \lambda\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}, \qquad \lambda = \frac{7GJ}{\omega^2 LI_D}. \tag{b}$$

Let the first arbitrary vector $\{\theta\}_1$ have all its elements unity and perform the multiplication

$$\begin{bmatrix} 5 & 3 & 2 \\ 3 & 6 & 4 \\ 1 & 2 & 6 \end{bmatrix}\begin{Bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{Bmatrix} = \begin{Bmatrix} 10.0000 \\ 13.0000 \\ 9.0000 \end{Bmatrix} = 10.0000\begin{Bmatrix} 1.0000 \\ 1.3000 \\ 0.9000 \end{Bmatrix}$$

Next we use the resulting vector as an improved trial vector and perform the second iteration,

$$\begin{bmatrix} 5 & 3 & 2 \\ 3 & 6 & 4 \\ 1 & 2 & 6 \end{bmatrix}\begin{Bmatrix} 1.0000 \\ 1.3000 \\ 0.9000 \end{Bmatrix} = \begin{Bmatrix} 10.7000 \\ 14.4000 \\ 9.0000 \end{Bmatrix} = 10.7000\begin{Bmatrix} 1.0000 \\ 1.3458 \\ 0.8411 \end{Bmatrix}.$$

The seventh iteration gives

$$\begin{bmatrix} 5 & 3 & 2 \\ 3 & 6 & 4 \\ 1 & 2 & 6 \end{bmatrix}\begin{Bmatrix} 1.0000 \\ 1.3413 \\ 0.7985 \end{Bmatrix} = \begin{Bmatrix} 10.6209 \\ 14.2418 \\ 8.4736 \end{Bmatrix} = 10.6209\begin{Bmatrix} 1.0000 \\ 1.3409 \\ 0.7978 \end{Bmatrix}.$$

At this point it is intuitively felt that one additional iteration will not produce any appreciable change, and we take as the first eigenvector and first eigenvalue,

$$\{\theta^{(1)}\} = \begin{Bmatrix} 1.0000 \\ 1.3409 \\ 0.7978 \end{Bmatrix}, \qquad \lambda_1 = 10.6209. \tag{c}$$

The second eigenvector must be orthogonal to the first,

$$\{\theta\}^T[I_p]\{\theta^{(1)}\} = I_D\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}^T\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} 1.0000 \\ 1.3409 \\ 0.7978 \end{Bmatrix} = 0, \tag{d}$$

which represents a constraint equation, allowing us to express θ_1 in terms of θ_2 and θ_3 as follows:

$$\theta_1 = -1.3409\theta_2 - 1.5956\theta_3, \tag{e}$$

and in addition we take arbitrarily

$$\theta_2 = \theta_2, \qquad \theta_3 = \theta_3. \tag{f}$$

Equations (e) and (f) are used to construct the first sweeping matrix,

$$[S^{(1)}] = \begin{bmatrix} 0 & -1.3409 & -1.5956 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{g}$$

The iteration to the second eigenvector must use the new dynamical matrix

$$\begin{aligned} [D^{(2)}] = [D][S^{(1)}] &= \frac{LI_D}{7GJ}\begin{bmatrix} 5 & 3 & 2 \\ 3 & 6 & 4 \\ 1 & 2 & 6 \end{bmatrix}\begin{bmatrix} 0 & -1.3409 & -1.5956 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \frac{LI_D}{7GJ}\begin{bmatrix} 0 & -3.7045 & -5.9780 \\ 0 & 1.9773 & -0.7868 \\ 0 & 0.6591 & 4.4044 \end{bmatrix}. \end{aligned} \tag{h}$$

We choose an arbitrary vector and write the first iteration for the second eigenvector,

$$\begin{bmatrix} 0 & -3.7045 & -5.9780 \\ 0 & 1.9773 & -0.7868 \\ 0 & 0.6591 & 4.4044 \end{bmatrix}\begin{Bmatrix} 0.0000 \\ 0.0000 \\ -1.0000 \end{Bmatrix} = \begin{Bmatrix} 5.9780 \\ 0.7868 \\ -4.4044 \end{Bmatrix} = 5.9780\begin{Bmatrix} 1.0000 \\ 0.1316 \\ -0.7368 \end{Bmatrix}.$$

The eleventh iteration yields

$$\begin{bmatrix} 0 & -3.7045 & -5.9780 \\ 0 & 1.9773 & -0.7868 \\ 0 & 0.6591 & 4.4044 \end{bmatrix}\begin{Bmatrix} 1.0000 \\ 0.3214 \\ -0.8962 \end{Bmatrix} = \begin{Bmatrix} 4.1669 \\ 1.3406 \\ -3.7354 \end{Bmatrix} = 4.1669\begin{Bmatrix} 1.0000 \\ 0.3217 \\ -0.8964 \end{Bmatrix},$$

at which point the iteration process was stopped. Hence we have the second eigenvector and the second eigenvalue,

$$\{\theta^{(2)}\} = \begin{Bmatrix} 1.0000 \\ 0.3217 \\ -0.8964 \end{Bmatrix}, \qquad \lambda_2 = 4.1669. \tag{i}$$

The third eigenvector must be orthogonal to the first two eigenvectors:

$$\{\theta\}^T[I_p]\{\theta^{(1)}\} = I_D\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 1.0000 \\ 1.3409 \\ 0.7978 \end{Bmatrix} = 0,$$

$$\{\theta\}^T[I_p]\{\theta^{(2)}\} = I_D\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} 1.0000 \\ 0.3217 \\ -0.8964 \end{Bmatrix} = 0. \tag{j}$$

These two simultaneous equations, when solved for θ_1 and θ_2 in terms of θ_3, give

$$\theta_1 = 2.8623\theta_3, \qquad \theta_2 = -3.3246\theta_3, \tag{k}$$

and arbitrarily taking

$$\theta_3 = \theta_3, \tag{l}$$

we can construct a new sweeping matrix

$$[S^{(2)}] = \begin{bmatrix} 0 & 0 & 2.8623 \\ 0 & 0 & -3.3246 \\ 0 & 0 & 1 \end{bmatrix}. \tag{m}$$

The dynamical matrix to be used for the iteration to the third eigenvector is

$$[D^{(3)}] = [D][S^{(2)}] = \frac{LI_D}{7GJ}\begin{bmatrix} 5 & 3 & 2 \\ 3 & 6 & 4 \\ 1 & 2 & 6 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2.8623 \\ 0 & 0 & -3.3246 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{LI_D}{7GJ}\begin{bmatrix} 0 & 0 & 6.3377 \\ 0 & 0 & -7.3607 \\ 0 & 0 & 2.2131 \end{bmatrix}. \tag{n}$$

Upon arbitrarily choosing a trial vector we iterate to the third eigenvector as follows:

$$\begin{bmatrix} 0 & 0 & 6.3377 \\ 0 & 0 & -7.3607 \\ 0 & 0 & 2.2131 \end{bmatrix}\begin{Bmatrix} 0.0000 \\ 0.0000 \\ 1.0000 \end{Bmatrix} = \begin{Bmatrix} 6.3377 \\ -7.3607 \\ 2.2131 \end{Bmatrix} = 6.3377\begin{Bmatrix} 1.0000 \\ -1.1614 \\ 0.3492 \end{Bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 6.3377 \\ 0 & 0 & -7.3607 \\ 0 & 0 & 2.2131 \end{bmatrix}\begin{Bmatrix} 1.0000 \\ -1.1614 \\ 0.3492 \end{Bmatrix} = \begin{Bmatrix} 2.2131 \\ -2.5704 \\ 0.7728 \end{Bmatrix} = 2.2131\begin{Bmatrix} 1.0000 \\ -1.1614 \\ 0.3492 \end{Bmatrix}.$$

It is easy to see that to obtain the third eigenvector there is not really any iteration process involved, because the nonzero column of the dynamical matrix $[D^{(3)}]$ is proportional to the last mode. The third eigenvector and eigenvalues are

$$\{\theta^{(3)}\} = \begin{Bmatrix} 1.0000 \\ -1.1614 \\ 0.3492 \end{Bmatrix}, \qquad \lambda_3 = 2.2131. \tag{o}$$

The eigenvectors can be normalized by writing

$$\{\theta^{(r)}\}^T[I_p]\{\theta^{(r)}\} = 1, \tag{p}$$

and the natural frequencies can be evaluated by using the second of equations (b). The normal modes and the natural frequencies are

$$\{\theta^{(1)}\} = I_D^{-1/2}\begin{Bmatrix} 0.4956 \\ 0.6646 \\ 0.3954 \end{Bmatrix}, \qquad \omega_1 = 0.8118\sqrt{\frac{GJ}{LI_D}},$$

$$\{\theta^{(2)}\} = I_D^{-1/2}\begin{Bmatrix} 0.6075 \\ 0.1954 \\ -0.5445 \end{Bmatrix}, \qquad \omega_2 = 1.2961\sqrt{\frac{GJ}{LI_D}}, \tag{q}$$

$$\{\theta^{(3)}\} = I_D^{-1/2}\begin{Bmatrix} 0.6210 \\ -0.7213 \\ 0.2169 \end{Bmatrix}, \qquad \omega_3 = 1.7784\sqrt{\frac{GJ}{LI_D}}.$$

The above results compare reasonably well with the results obtained by using the characteristic determinant method.